THE MOTION OF FOUR RECTILINEAR VORTEX FILAMENTS

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1. INTRODUCTION

The motion of three parallel rectilinear vortex filaments of strengths K_1, K_2 K_3 in a perfect incompressible fluid extending to infinity originally considered by Grobli (1877) has been again considered by Synge (1949) who has made an exhaustive study of the motion by representing the various configurations of vortices by points in trilinear co-ordinates on a representative plane. A similar study of all the motions of four vortex filaments presents great difficulty owing to the very large number of cases to be dealt with and the lack of corresponding geometrical representation adequate for classifying all the motions. The problem of four vortex filaments which form fixed configurations, i.e., configurations with fixed sides and diagonals was previously examined(1951). Synge's method can however be used for the study of motions of four vortex filaments by making some simplifying assumptions. We study in particular the case where the configuration is initially a parallelogram and the vortices at the ends of diagonals are of equal strengths.

By taking the adjacent sides and a diagonal which completely specify the configuration as the geometrical magnitudes, it is possible to classify all the motions in a manner similar to Synge's and the present paper deals with this problem. The theory resembles that of the motion of three vortices in many respects, but it is felt that it may be of interest to see the changes due to the presence of an additional vortex. Under the more particular case of rectangular configuration with special choice of vortex strengths, the motion is examined by representing the configurations by cartesian co-ordinates.

2. EQUATIONS OF MOTION

Vortices of strengths K_1, K_2, K_1, K_2 lie initially at the corners A, B, C, D of a parallelogram. Let a, c and b, g be the adjacent sides and diagonals respectively, the diagonal b joining the vortices K_1, K_1 and g joining K_2, K_2. Then these vortices remain at the corners of a parallelogram at the end of any time (Gorjatschaff, 1898). The configuration is uniquely determined by the magnitudes a, b, c except for orientation and the motion of the vortices is represented by a point on a representative plane by trilinear co-ordinates.

The following equations of motion of the vortices have been obtained by Lakshmana Rao (1951).

$$\frac{da}{dt} = S a^{-1}\left[K_1\left(b^{-2}-c^{-2}\right) - K_2\left(g^{-2}-c^{-2}\right)\right] \tag{2.1}$$

$$\frac{db}{dt} = S b^{-1}\left[2K_2\left(c^{-2}-a^{-2}\right)\right] \tag{2.2}$$

$$\frac{dc}{dt} = S c^{-1}\left[K_1\left(a^{-2}-b^{-2}\right) - K_2\left(a^{-2}-g^{-2}\right)\right] \tag{2.3}$$

where S = area of the parallelogram ABCD.

We have the integrals due to Kirchhoff (Lamb, 1945).

$$2K_1K_2 \log ac + K_1^2 \log b + K_2^2 \log g = \text{const.} = \alpha \tag{2.4}$$

$$K_1 b^2 + K_2 g^2 = \text{const.} = \beta . \tag{2.5}$$

3. VARIABLE CONFIGURATIONS

Following Synge's procedure we write

$$x_1 = a/(a+b+c) \tag{3.1}$$

$$x_2 = b/(a+b+c) \tag{3.2}$$

$$x_3 = c/(a+b+c) \tag{3.3}$$

and take x_1, x_2, x_3 to signify the trilinear co-ordinates of a point with an equilateral triangle of unit altitude for the triangle of reference so that

$$x_1 + x_2 + x_3 = 1 . \tag{3.4}$$

For any parallelogram configuration we have definite values of a, b, c giving a unique representative point x_1, x_2, x_3. For a given x-point we have an infinity of similar parallelogram configurations. As in Synge's theory of three vortices, the possible collinear configurations correspond to points on the lines of join of the middle points of the reference triangle. Essentially there are two distinct collinear configurations only. Differentiation of the equations (3 1), (3 2),(3 3) gives

$$\frac{dx_1}{dt} = \frac{da}{dt}(a+b+c)^{-1} - a(a+b+c)^{-2}\left(\frac{da}{dt}+\frac{db}{dt}+\frac{dc}{dt}\right)$$

and two similar equations. It is seen that

$$\frac{dx_1}{dt} = KG_1, \quad \frac{dx_2}{dt} = KG_2, \quad \frac{dx_3}{dt} = KG_3$$

with

$$K = S(a+b+c)^4(abcg)^{-2}, \tag{3.5}$$

$$G_1 = -K_1x_1(x_2^2-x_3^2)x^2 + K_2x_1(x^2-x_3^2)x_2^2 + x_1\theta, \tag{3.6}$$

$$G_2 = -2K_2x_2(x_3^2-x_1^2)x^2 + x_2\theta, \tag{3.7}$$

$$G_3 = -K_1x_3(x_1^2-x_2^2)x^2 + K_2x_3(x_1^2-x^2)x_2^2 + x_3\theta \tag{3.8}$$

and

$$\theta = K_1x_1(x_2^2-x_3^2)x^2 + 2K_2x_2(x_3^2-x_1^2)x^2 + K_1x_3(x_1^2-x_2^2)x^2$$
$$- K_2x_1(x^2-x_3^2)x_2^2 - K_2x_3(x_1^2-x^2)x_2^2, \tag{3.9}$$

where

$$g/(a+b+c) = x, \tag{3.10}$$

so that

$$x^2 = 2x_1^2 - x_2^2 + 2x_3^2. \tag{3.11}$$

The differential equations

$$\frac{dx_1}{G_1} = \frac{dx_2}{G_2} = \frac{dx_3}{G_3} = K\,dt \tag{3.12}$$

give the motion of the representative point. They define a congruence of x-curves giving the behaviour of the configurations and also the rate of change while the size of the configuration is given by one of the integrals (2·4), (2·5).

The following results are then simply deduced.

If the strengths of the vortices do not satisfy the relation

$$K_1^2 + K_2^2 + 4K_1K_2 = 0 \tag{3·13}$$

and α is known from the initial configuration (equation 2·4) to each x-point there corresponds a unique parallelogram configuration except for orientation. Whereas if they satisfy the relation (3·13) then to each point on the conic (hyperbola)

$$K_1 x_2^2 + K_2 x^2 = 0 \tag{3·14}$$

there corresponds a single infinity of parallelogram configurations ob both orientations, and to a point off this conic there corresponds a unique parallelogram configuration except for orientation.

4. SINGULAR POINTS

The equations

$$G_1 = -K_1 x_1 (x_2^2 - x_3^2) x^2 + K_2 x_1 (x^2 - x_3^2) x_2^2 + x_1 \theta = 0, \tag{4·1}$$

$$G_2 = -2K_2 x_2 (x_3^2 - x_1^2) x^2 + x_2 \theta = 0, \tag{4·2}$$

$$G_3 = -K_1 x_3 (x_1^2 - x_2^2) x^2 + K_2 x_3 (x_1^2 - x^2) x_2^2 + x_3 \theta = 0 \tag{4·3}$$

give the singular points of the representation curves. Under the condition $K_1 + K_2 = 0$ the collisions of unequal vortices, viz., the cases $x_1 = 0, x_2 = x_3 = \frac{1}{2}$ and $x_3 = 0, \; x_1 = x_2 = \frac{1}{2}$ are seen to correspond to singular points.

To see any other singular points we first consider the case where $\theta = 0$. Then from (4.2) we see that $x_1 = x_3$ and additionally (4·1) or (4·3) gives

$$K_1 (x_2^2 - x_1^2) x^2 - K_2 (x^2 - x_1^2) x_2^2 = 0 \tag{4·4}$$

The case $x_1 = x_2 = x_3$ is seen to be exceptional and is to be ruled out.

In the case $\theta \neq 0$ we have $x_1 \neq x_3$. We h ve then the easily obtainable relation

$$\theta\left(K_1^2 + 4K_1K_2 + K_2^2\right) = 2K_2\left(x_3^2 - x_1^2\right)\left(K_1 x_2^2 + K_2 x^2\right) \qquad (4\cdot 5)$$

In case equation (3·13) holds we hare ledto equation (3·14)

$$K_1 x_2^2 + K_2 x^2 = 0 . \qquad ([illegible])$$

We have thus the results:

If

$$K_1^2 + K_2^2 + 4K_1K_2 \neq 0 . \qquad (4\cdot 6)$$

and

$$K_1 + K_2 \neq 0 \qquad (4\cdot 7)$$

the singular points arise when $x_1 = x_3$ corresponding to rhombus configurations. The case $x_1 = x_2 = x_3$ corresponding to an angle 60° in the rhombus is an exceptional case.

If

$$K_1^2 + K_2^2 + 4K_1K_2 = 0 \qquad (4\cdot 8)$$

we necessarily have (4·7) and the singular points lie on the hyperbola

$$K_1 x_2^2 + K_2 x^2 = 0 .$$

If

$$K_1 + K_2 = 0 \qquad (4\cdot 9)$$

(4·6) is necessarily true and the only singular points are when $x_1 = 0$, $x_2 = x_3 = \frac{1}{2}$ or $x_3 = 0$, $x_1 = x_2 = \frac{1}{2}$ corresponding to collisions in pairs of unequal vortices.

5. RECTANGULAR CONFIGURATION

If vortices of strengths $K, -K, K, -K$ lie at the corners of a rectangle initially, they do so at the end of any time (Lamb, 1945) and the configuration can never pass through collinearity. The initial orientation is thus preserved for all time.

To study the motion we may write

$$a = x, \quad c = y \qquad\qquad 0 < x, y < \infty$$

and take (x, y) as the rectangular Cartesian co-ordinates of a point on a representative plane. The co-ordinate axes are non-reachable barriers for the motion of (x, y).

The equations of motion are easily seen from (2·1), (2·3) to be

$$-\frac{dx}{x^3} = \frac{dy}{y^3} = \frac{2K}{xy(x^2+y^2)}\,dt.$$

The representative point moves on the curve

$$x^{-2} + y^{-2} = \text{const.} = 1/d^2.$$

which is equivalent to the integral (2·4).

This congruence of curves has no singularities and the configuration cannot remain fixed.

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ON A CLASS OF VISCOUS COMPRESSIBLE FLOWS

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ON A CLASS OF VISCOUS COMPRESSIBLE FLOWS

Intoduction. The equation of motion of a viscous compressible fluid can be put in the form[1]

$$(1) \quad \frac{d\vec{q}}{dt} = -\nabla\left(\Omega + \int \frac{dp}{\rho}\right) + \frac{\nu}{3}\nabla(\nabla\vec{q}) + \nu\nabla^2\vec{q}$$

when the external force field is conservative and is derivable from the potential Ω. $\vec{q}$, ρ, p, ν are respectively the fluid velocity vector, the density, the internal stress or pressure and the kinematic viscosity. The equation can be integrated once if ν is constant and the motion is irrotational, the integral being

$$(2) \quad -\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + \Omega + \int\frac{dp}{\rho} - \frac{4}{3}\nu\nabla\vec{q} = F(t)$$

where ϕ is the velocity potential and $\vec{q} = -\nabla\phi$. We can integrate the equation the equation of motion also when the flow is steady and the fluid is inviscid, the result being the wellknown Bernoulli's theorem. Inviscid fluid motion is further characterized by the constancy of circulation and the permanence of rotational or irrotational nature of the flow. In a viscous incompressible fluid however, the circulation varies at a rate depending on the kinematic viscosity and the space rates of change of vorticity components. The vorticity itself experiances a decay on account of viscosity.

In the present note we consider the class of flows of viscous compressible fluids for which the vector $\nabla^2\vec{q}$ is irrotational and has the scalar potential function H. It is shown here that such flows are similar to an inviscid flow in general characteristics. The kinematic viscosity is assumed constant.

For the flows under consideration

$$(3) \quad \nabla^2\vec{q} = -\nabla H;$$

the equation of motion can therefore be written as

$$\frac{d\vec{q}}{dt} = -\nabla\left(\Omega + \int\frac{dp}{\rho}\right) + \frac{1}{3}\nu\nabla(\nabla\vec{q}) - \nu\nabla H$$

or in the form

(4) $$\frac{\partial \vec{q}}{\partial t} - \vec{q} \times \vec{\zeta} = -\nabla\left(\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{1}{3}\nu\,\nabla\vec{q} + \nu H\right)$$

where $\vec{\zeta} = \text{curl } \vec{q}$. For steady flow $\frac{\partial \vec{q}}{\partial t} = 0$ and

(4') $$\vec{q} \times \vec{\zeta} = \nabla\left(\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{1}{3}\nu\,\nabla\vec{q} + \nu H\right)$$

Scalar multiplication of the two sides of (4') by $\frac{\vec{q}}{|\vec{q}|}$ and $\frac{\vec{\zeta}}{|\vec{\zeta}|}$ gives

$$\frac{\partial}{\partial s}\left(\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{1}{3}\nu\,\nabla\vec{q} + \nu H\right) = 0$$

and

$$\frac{\partial}{\partial \sigma}\left(\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{1}{3}\nu\,\nabla\vec{q} + \nu H\right) = 0$$

ds, $d\sigma$ being elements of arc along streamlines and vortex lines respectively. We have therfore

(5) $$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{1}{3}\nu\,\nabla\vec{q} + \nu H = \text{constant}$$

along streamlines as well as vortex lines. This result, insofar as it relates to streamlines, is an extension of Bernoulli's theorem and its analogue for plane incompressible flows has been noticed by Gortler and Wieghardt[2].

The circulation in a closed circuit in the fluid is $c = \int \vec{q}\cdot d\vec{r}$

$$\begin{aligned}\frac{dc}{dt} &= \int \frac{d\vec{q}}{dt}\cdot d\vec{r} + \int \vec{q}\cdot\frac{d}{dt}(d\vec{r})\\ &= -\int \nabla\left(\Omega + \int\frac{dp}{\rho} - \frac{\nu}{3}\nabla\vec{q} + \nu H\right)\cdot d\vec{r} + \int \vec{q}\cdot d\vec{q}\\ &= -\int \frac{\partial}{\partial s}\left(\Omega + \int\frac{dp}{\rho} - \frac{\nu}{3}\nabla\vec{q} + \nu H\right)ds + \int\frac{\partial}{\partial s}\left(\frac{1}{2}q^2\right)ds = 0.\end{aligned}$$

The ciculation in any circuit moving with the fluid is constant as in inviscid fluid motion.

The curl of the twq sides of equation (4) gives

$$\frac{\partial}{\partial t}\left(\text{curl }\vec{q}\right) - \text{curl}\left(\vec{q}\times\vec{\zeta}\right) = 0$$

or

$$\frac{\partial}{\partial t}\left(\text{curl }\vec{q}\right) - \left\{\left(\text{curl }\vec{q}\cdot\nabla\right)\vec{q} - \left(\vec{q}\cdot\nabla\right)\text{curl }\vec{q} - \left(\text{curl }\vec{q}\right)\left(\text{div }\vec{q}\right) + \vec{q}\,\text{div}\left(\text{curl }\vec{q}\right)\right\} = 0$$

(6) i.e., $$\frac{d}{dt}\left(\text{curl }\vec{q}\right) = \left(\text{curl }\vec{q}\cdot\nabla\right)\vec{q} - \left(\text{curl }\vec{q}\right)\left(\text{div }\vec{q}\right).$$

Helmholtz's theorem in inviscid fluids can be put in the form[3]

$$\frac{d}{dt}\left(\frac{\vec{\zeta}}{\rho}\right) = \frac{1}{\rho}\left(\vec{\zeta}\cdot\nabla\right)\vec{q}$$

and (6) is equivalent to it. (6) thus provides an extension of Helmholtz's theorem. A consequence of (6) is that the rotational nature of any portion of the fluid is permanent as in inviscid fluids. This is evident otherwise, for in the class of flows considered here, there is no decay of vorticity in spite of viscosity, as $\nabla^2\vec{\zeta}=0$.

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THE VELOCITY OF A ROTATIONAL VISCOUS FLOW INSIDE A FIXED CONTAINER

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We express in this note the velocity of a viscous fluid motion within a fixed container in terms of the expansion and vorticity. We show that the velocity is given by

$$(1) \quad \vec{q}_P = -\frac{\partial}{\partial \vec{P}} \frac{1}{4\pi}\int_{(V)} \frac{\theta_Q}{PQ}\, d\tau + \frac{\partial}{\partial \vec{P}} \times \frac{1}{4\pi}\int_{(V)} \frac{\vec{\zeta}_Q}{PQ}\, d\tau$$

where E is the surface of the container enclosing the volume V and θ, $\vec{\zeta}$ are the expansion and vorticity supposed to be given at every point of V. $\frac{\partial}{\partial \vec{P}}$ denotes the operator ∇, whenever differentiations are with respect to the co-ordinates of P. The relation (1) incidentally provides an irrotational-solenoidal resolution of the velocity vector field $\vec{q}$ in the finite region (V) for we can write from (1)

$$(2) \quad \vec{q} = -\operatorname{grad}\left(\frac{1}{4\pi}\int_{(V)} \frac{\theta}{r}\, d\tau\right) + \operatorname{curl}\left(\frac{1}{4\pi}\int_{(V)} \frac{\vec{\zeta}}{r}\, d\tau\right)$$

at all points of (V).

Proof of relation (1): θ and $\vec{\zeta}$ are defined everywhere in (V) and we have the boundary condition $\vec{q}=0$ on (E). The vector function

$$(3) \quad \vec{A}_P = \frac{1}{4\pi}\int_{(V)} \frac{\vec{q}_Q}{PQ}\, d\tau$$

satisfies Poisson's equation

$$\nabla^2 \vec{A}_P = \frac{\partial^2 \vec{A}_P}{\partial \vec{P}^2} = -\vec{q}_P ,$$

so that

$$(4) \quad \vec{q}_P = -\nabla^2 \vec{A}_P = \frac{\partial}{\partial \vec{P}} \times \left(\frac{\partial}{\partial \vec{P}} \times \vec{A}_P\right) - \frac{\partial}{\partial \vec{P}}\left(\frac{\partial \vec{A}_P}{\partial \vec{P}}\right).$$

Now

$$\frac{\partial}{\partial \vec{P}} \times \left(\vec{A}_P\right) = \frac{1}{4\pi}\frac{\partial}{\partial \vec{P}} \times \int_{(V)} \frac{\vec{q}_Q}{PQ}\, d\tau = -\frac{1}{4\pi}\int_{(V)} \vec{q}_Q \times \frac{\partial}{\partial \vec{P}}\left(\frac{1}{PQ}\right) d\tau$$

$$= \frac{1}{4\pi}\int_{(V)} \vec{q}_Q \times \frac{\partial}{\partial \vec{P}}\left(\frac{1}{PQ}\right) d\tau = \frac{1}{4\pi}\int_{(V)} \frac{1}{PQ}\left(\frac{\partial}{\partial \vec{Q}} \times \vec{q}_Q\right) d\tau - \frac{1}{4\pi}\int_{(V)} \frac{\partial}{\partial \vec{Q}} \times \left(\frac{\vec{q}_Q}{PQ}\right) d\tau$$

$$= \frac{1}{4\pi}\int_{(V)} \frac{\vec{\zeta}_Q}{PQ}\, d\tau \;+\; \frac{1}{4\pi}\int_{(E)} \frac{\vec{n}\times\vec{q}_Q}{PQ}\, dS$$

by Gauss' theorem, $\vec{n}$ being the unit vector along the inward normal to the surface (E), and here the surface integral vanishes by the boundary condition. Hence

$$(5) \qquad \frac{\partial}{\partial\vec{P}}\times\vec{A}_P \;=\; \frac{1}{4\pi}\int_{(V)}\frac{\vec{\zeta}_Q}{PQ}\, d\tau$$

Also

$$\frac{\partial\vec{A}_P}{\partial\vec{P}} \;=\; \frac{1}{4\pi}\int_{(V)}\vec{q}_Q\cdot\frac{\partial}{\partial\vec{P}}\left(\frac{1}{PQ}\right)d\tau \;=\; -\,\frac{1}{4\pi}\int_{(V)}\vec{q}_Q\cdot\frac{\partial}{\partial\vec{Q}}\left(\frac{1}{PQ}\right)d\tau$$

$$= \cdot\;\cdot \;=\; \frac{1}{4\pi}\int_{(V)}\frac{\theta_Q}{PQ}\,d\tau \;+\; \frac{1}{4\pi}\int_{(E)}\left(\vec{n}\cdot\vec{q}_Q/PQ\right)dS$$

again by Gauss' therem and the surface integral vanishes as before, so that

$$(6) \qquad \frac{\partial\vec{A}_P}{\partial\vec{P}} \;=\; \frac{1}{4\pi}\int_{(V)}\frac{\theta_Q}{PQ}\,d\tau$$

Using (5) and (6) in (4) we obtain the result (1).

Synge* has obtained the necessary and sufficient condition for the consistence of given expansion and vorticity in a fixed container with vanishing velocity on the boundary in the form

$$(7) \qquad \int_{(V)}\left(\theta P + \vec{\zeta}\cdot\vec{Q}\right)d\tau \;=\; 0$$

where $\vec{Q}$ is an arbitrary sollution of $\nabla^2\vec{Q} = \mathrm{grad}\left(\mathrm{div}\,\vec{Q}\right)$, and P is given by the equation grad $\vec{P} = \frac{1}{2}\mathrm{curl}\;\vec{Q}$. We find on taking the divergense of (5)

$$(8) \qquad 0 \;=\; \frac{\partial}{\partial\vec{P}}\int_{(V)}\frac{\vec{\zeta}_Q}{PQ}\,d\tau \;=\; -\int_{(V)}\vec{\zeta}_Q\cdot\frac{\partial}{\partial\vec{Q}}\left(\frac{1}{PQ}\right)d\tau$$

This can be compared with Synge's condition (7) by taking $\vec{Q} = \mathrm{grad}\,\frac{1}{r}$ and $P = 0$.

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TRANSPORT OF VORTICITY IN A COMPRESSIBLE FLOW

TRANSPORT OF VORTICITY IN A COMPRESSIBLE FLOW

In a viscous compressible flow transport of vorticity is brought about by convection as well as diffusion. The rate of transport of vorticity through an open surface (S) in a fluid is

$$(1)\quad \vec{T}_S = \int_{(S)} (\vec{n}\cdot\vec{q})\,\vec{\zeta}\,dS \;-\; \int_{(S)} \nu \frac{\partial \vec{\zeta}}{\partial n}\,dS$$

$\vec{q}$, $\vec{\zeta}$ are the velocity and vorticity vectors, $\vec{n}$ is a unit vector along the outward normal to S and ν is the (constant) kinematic viscosity of the fluid.

If S is a vortex diaphragm, i.e., an open surface consisting of vortex lines and closing a vortex line(C) we have

Preston's Theorem[1]: In the steady motion of a homogeneous liquid of uniform viscosity the rate of transport of vorticity through a vortex diaphragm closing the vortex line C is

$$(2)\quad \vec{T}_S = \int_{(C)} \left(\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega\right) d\vec{s}\ .$$

It is significant that $\vec{T}_S$ does not involve ν explicitly.

In a steady flow of a compressible fluid of constant kinematic viscosity, the equation of motion (with the usual assumptions and notation) is

$$(3)\quad \vec{q}\times\vec{\zeta} = \nabla\left(\Omega + \int\frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{4}{3}\nu\,\nabla\vec{q}\right) + \nu\,\nabla\times\vec{\zeta}$$

Hence $\vec{n}\times(\vec{q}\times\vec{\zeta}) = \vec{n}\times\nabla\left(\Omega + \int\frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{4}{3}\nu\,\nabla\vec{q}\right) + \nu\,\vec{n}\times(\nabla\times\vec{\zeta})$

$$(4)\ \text{i.e.,}\quad (\vec{n}\cdot\vec{q})\,\vec{\zeta} - \nu\frac{\partial\vec{\zeta}}{\partial n} = -\vec{n}\times\nabla\left(\Omega + \int\frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{4}{3}\nu\,\nabla\vec{q}\right) - \nu\,(\vec{n}\times\nabla)\times\vec{\zeta}.$$

On integrating this equation and using Stokes' theorem ~~wak~~ we have

$$\vec{T}_S = -\int_{(C)}\left(\Omega + \int\frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{4}{3}\nu\,\nabla\vec{q}\right) d\vec{s}$$

which is Preston's formula for compressible flows.

[1] Milne-Thomson, L. M. Theoretical Hydrodynamics (Second Edition) 1949. p. 527.

ON THE PERMANENCE OF VECTOR-LINES IN A RIEMANNIAN V_n

OON THE PERMANENCE OF VECTOR-LINES IN A RIEMANNIAN V_n

The condition for the permanence of vector-lines in a moving fluid due to Zorawski has been proved by arguments of Vector Analysis some time ago by Prim and Truesdell[1]. Synge[2] has obtained the condition for permanence of vector-lines in a Euclidean space of N dimensions adopting a slightly different approach. The present note is intended only to remark that Synge's approach holds true even for a general Riemannian space V_n. The changes in Synge's proof needed for this more general case are only formal.

V_n is a Riemannian space with the fundamental form $g_{ij}\, dx^i\, dx^j$. $v^i(x^1, \ldots, x^n; t)$ and $c^i(x^1, \ldots, x^n; t)$ are two (contravariant) vector fields containing a parameter t which may be taken to represent the time. v^i is the primary field and plays the role of velocity. The vector-lines of the secondary field c^i are the curves satisfying the equation

$$(1) \qquad c^i\, dx^j - c^j\, dx^i = 0 .$$

We limit ourselves by observing that all the equations in Synge's paper hold true for a V_n when appropriate changes are made in the positions of indices. Thus, for example, we have

$$(2) \qquad x^i = f^i(\theta, t)$$

where θ is a parameter remaining constant for a fluid particle and

$$(3) \qquad \frac{dx^i}{dt} = \frac{\partial f^i}{\partial t} = v^i , \qquad \frac{\partial f^i}{\partial \theta} = \lambda^i$$

The condition that a curve C which is a vector-line of the secondary field c^i at a certain instant $t = t_0$ continues to do so for all time is that the contravariant tensor

$$(4) \qquad \Omega^{ij} = c^i \lambda^j - c^j \lambda^i$$

has zero components.

We may easily see that

$$(5) \qquad \frac{\partial \Omega^{ij}}{\partial t} = \lambda^{l} \left(A^{ij}_{l} - A^{ji}_{l} \right)$$

where $A^{ij}_{l} = \delta^{j}_{l} \frac{\partial c^{i}}{\partial t} + \delta^{j}_{l} c^{i}_{,k} v^{k} + c^{i} v^{j}_{,l}$.

The comma denotes covariant differentiation and the summation convention is used. If C is a vector-line at $t = t_0$ so that $\Omega^{ij} = 0$ at $t = t_0$, it is necessary and sufficient for the permanence of it as a vector-line that $\frac{\partial \Omega^{ij}}{\partial t} = 0$.
Hence the necessary and sufficient condition for the permanence of vector-lines in a V_n is that the tensor $c^{l} A^{ij}_{l}$ should be symmetric.

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SUPPORTING PAPERS

AXISYMMETRIC FLUID MOTIONS WITH A COMMON FLOW PATTERN

AXISYMMETRIC FLUID MOTIONS WITH A COMMON FLOW PATTERN

Specification of a fluid motion means a knowledge of the streamline pattern as well as the magnitude of the flow velocity at all points. However the flow pattern and the velocity are interdependent though distinct features of a flow and it is interesting to examine the measure of this interdependence in some way. Gilbarg[1] has pointed out a way to this end by classifying the set of all dynamically distinct plane flows of inviscid liquids having the same streamline pattern and this has led to a further study of the uniqueness of flows with given streamlines[2,3]. In the present note we classify all axisymmetric motions of inviscid liquids with the same flow pattern.

Steady axisymmetric flow of a non-viscous incompressible fluid is defined in a region of the meridian plane by the Stokes' stream function $\psi(x,\varpi)$ where x and ϖ are coordinates in the axial and radial directions. If $\vec{q}$, ρ, and p are the fluid velocity, fluid density, and pressure the equations of motion, when the external forces are absent, are given by

$$q_x \frac{\partial q_x}{\partial x} + q_\varpi \frac{\partial q_x}{\partial \varpi} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad q_x \frac{\partial q_\varpi}{\partial x} + q_\varpi \frac{\partial q_\varpi}{\partial \varpi} = -\frac{1}{\rho}\frac{\partial p}{\partial \varpi}$$

or, since $q_x = -\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}$, $q_\varpi = \frac{1}{\varpi}\frac{\partial \psi}{\partial x}$ also by the equations

$$(1)\quad \begin{aligned} &\frac{1}{\varpi^2}\frac{\partial \psi}{\partial \varpi}\frac{\partial^2 \psi}{\partial x \partial \varpi} + \frac{1}{\varpi^3}\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial \varpi} - \frac{1}{\varpi^2}\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial \varpi^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \\ -&\frac{1}{\varpi^2}\frac{\partial \psi}{\partial \varpi}\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{\varpi^3}\left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{1}{\varpi^2}\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial \varpi \partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial \varpi}. \end{aligned}$$

If another such flow with stream function $\bar{\psi}(x,\varpi)$ has the same stream line pattern we have $\bar{\psi} = f(\psi)$. If $\bar{\rho}$ and $\bar{p}$ are the density and pressure of the latter flow, the equations of motion of this flow can be expressed in the form

$$(2)\quad [f'(\psi)]^2 \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}, \quad [f'(\psi)]^2 \frac{1}{\rho}\frac{\partial p}{\partial \varpi} = \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \varpi}.$$

Hence it follows that $\dfrac{\partial(p,\bar{p})}{\partial(x,\varpi)} = 0$. The compatibility relation for the equations (2) leads to

$$f'(\psi).\, f''(\psi)\cdot\frac{\partial(p,\psi)}{\partial(x,\varpi)} = 0 .$$

Hence $f'(\psi)$ is a constant so that $\bar{\psi} = a\psi + b$ where a and b are constants or $\frac{\partial(p, \bar{\psi})}{\partial(x, \varpi)} = 0$.* In the latter case we may easily see that $\frac{\partial(\bar{p}, \bar{\psi})}{\partial(x, \varpi)} = 0$. We have then the following result:

Steady axisymmetric flows of inviscid incompressible fluids with a common flow pattern have in general proportional velocity fields. The case of flows with the same streamline pattern but with velocity fields not proportional to one another corresponds to flows for which the lines of equal pressure (and hence the lines of equal magnitude of velocity i.e., the isovels) [illegible] coincide with the streamlines. In this case the function f is fairly arbitrary.

It is a known result[2] that in an axisymmetric motion of an inviscid liquid when the external forces are absent, the isovels coincide with the streamlines if and only if the flow is purely axial. We may therefore restate the latter part of the above result in the form:

Two axisymmetric flows with the same flow patterns can have velocity fields which are not proportional only in the trivial case where the flows are purely axial.

We may now examine the possible flow patterns common to potential and rotational axisymmetric motions of non-viscous liquids. If ψ is the stream function of a potential flow and $\bar{\psi} = f(\psi)$ that of any other flow we have

$$(3) \qquad \frac{1}{\varpi}\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{\varpi^2}\frac{\partial \psi}{\partial \varpi} + \frac{1}{\varpi}\frac{\partial^2 \psi}{\partial \varpi^2} = 0 ,$$

and

$$(3') \qquad \frac{1}{\varpi}\frac{\partial^2 \bar{\psi}}{\partial x^2} - \frac{1}{\varpi^2}\frac{\partial \bar{\psi}}{\partial \varpi} + \frac{1}{\varpi}\frac{\partial^2 \bar{\psi}}{\partial \varpi^2} = \frac{f''}{\varpi} \cdot \left(\psi_x^2 + \psi_\varpi^2\right) .$$

The flow corresponding to $\bar{\psi}$ is thus rotational only if $f'' \neq 0$ and then the flows are seen above to be purely axial. Thus we have the result:

The only flow patterns common to potential and rotational axisymmetric motions of inviscid liquids are those corresponding to purely axial flows.

It is of considerable interest to examine the flow patterns common to axisymmetric motions of incompressible and compressible fluids. Here we shall confine ourselves to noting a certain relation which may enable us to determine all the flow patterns common to steady, axisymmetric potential flows of liquids and of ideal gases.

If ϕ, ψ and $\bar{\phi}, \bar{\psi}$ are the velocity potentials and stream functions of the liquid and gas flows respectively, it is necessary and sufficient for these distinct types of flows to have the same flow pattern that $\bar{\phi} = f(\phi)$, $\bar{\psi} = g(\psi)$. We have the equations

(4) $\quad \dfrac{\partial \phi}{\partial x} = \dfrac{1}{\varpi}\dfrac{\partial \psi}{\partial \varpi}, \quad \dfrac{\partial \phi}{\partial \varpi} = -\dfrac{1}{\varpi}\dfrac{\partial \psi}{\partial x};$

(4') $\quad \dfrac{\partial \bar{\phi}}{\partial x} = \dfrac{1}{\rho\varpi}\dfrac{\partial \bar{\psi}}{\partial \varpi}, \quad \dfrac{\partial \bar{\phi}}{\partial \varpi} = -\dfrac{1}{\rho\varpi}\dfrac{\partial \bar{\psi}}{\partial x};$

where $\rho = \rho(x,\varpi)$ is the density function for the gas flow. A consequence of these two equations is

(5) $\quad f'(\phi) = \frac{1}{\rho}\, g'(\psi)$.

If $p = A\rho^{\gamma}$, from Bernoulli's theorem for the gas flow we derive the relation

(6) $\quad \rho = K\left(1 - f'^2 q^2\right)^{1/(\gamma-1)}$

where q is the magnitude of the (suitably normalized) velocity of the liquid flow and K can be taken as a reference constant of the gas flow. As with plane motions[1] we can deduce here again

(7) $\quad q^2 = (X - Y) \big/ X^{\frac{\gamma+1}{\gamma-1}}$

where $X = X(\phi) = (f')^{\gamma-1}$, and $Y = Y'(\psi) = (g')^{\gamma-1}$.

It is easily verified that a flow pattern corresponding to purely axial flows is common to axisymmetric potential motions of liquids and gases. Eq.(7) may be of use in finding out if there are other flow patterns common to potential motions of liquids and gases.

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A CLASS OF AXISYMMETRIC MOTIONS OF VISCOUS LIQUIDS

(To be published in Math. Zeitschrift)

A CLASS OF AXISYMMETRIC MOTIONS OF VISCOUS LIQUIDS

The motion of a viscous liquid subjected to a conservative external force field is governed by the Navier-Stokes differential equations

$$(1) \qquad \frac{d\vec{q}}{dt} = -\operatorname{grad}\left(\Omega + \frac{p}{\rho}\right) + \nu \nabla^2 \vec{q} \;,$$

$\vec{q}$ being the velocity vector. The continuity equation is

$$(1') \qquad \operatorname{div} \vec{q} = 0 \;.$$

We make the assumption that $\nabla^2\vec{q}$ is the gradient of a scalar function. Gortler and Wieghardt[1] have shown that steady plane flows of this class are characterized by the the biharmonicity of the stream function; i.e., the stream function of all such flows satisfy the equation $\nabla^4 \psi = \nabla^2(\nabla^2 \psi) = 0$.

In this note it will be shown that all steady, axisymmetric motions of viscous liquids, belonging to the above class, are characterized by the equation

$$E^4 \psi = E^2(E^2\psi) = 0 \;,$$

where $E^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial \varpi}$; x and ϖ are the co-ordinates in the axial and radial directions in a meridian plane and ψ is the (Stokes') stream function. The flow patterns common to (viscous) motions of this class and steady axisymmetric potential flows of inviscid liquids will also be examined.

Let

$$(2) \qquad \nabla^2 \vec{q} = -\nabla H \;.$$

In axisymmetric motions $(\vec{q})_x = -\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}$, $(\vec{q})_\varpi = \frac{1}{\varpi}\frac{\partial \psi}{\partial x}$.

For steady motion

$$(3) \qquad \vec{q} \times \operatorname{curl} \vec{q} = \operatorname{grad}\left(\Omega + \frac{p}{\rho} + \frac{1}{2}q^2 + \nu H\right).$$

Scalar multiplication of (3) with $\vec{q}/|\vec{q}|$ leads to

$$\frac{\partial}{\partial s}\left(\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega + \nu H\right) = 0 \;,$$

ds being an arc element along a streamline, so that

$$(4) \qquad \frac{p}{\rho} + \frac{1}{2}q^2 + \Omega + \nu H = h(\psi) \;.$$

From (2) and (3)

$$\nabla^2\left(-\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right) = -\frac{\partial H}{\partial x}\ , \quad \nabla^2\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial x}\right) = -\frac{\partial H}{\partial\varpi}$$

so that

$$(5)\qquad \frac{\partial}{\partial\varpi}\left[\nabla^2\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right)\right] + \frac{\partial}{\partial x}\left[\nabla^2\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial x}\right)\right] = 0 .$$

Writing $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\varpi^2} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}$, we may simplify (5) into

$$(6)\qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\right)^2\psi = 0 .$$

Noting that $\vec{q} \times \operatorname{curl}\vec{q} = \frac{E^2\psi}{\varpi^2}\operatorname{grad}\psi$ we obtain from (3) and (4)

$$(7)\qquad \frac{1}{\varpi^2}\left(E^2\psi\right)\operatorname{grad}\psi = h'(\psi)\operatorname{grad}\psi .$$

Now if $\operatorname{grad}\psi \neq 0$, we have $E^2\psi = \varpi^2 h'(\psi)$.

The hydrodynamical problem thus reduces to the solution of the equations

$$(8)\qquad E^2\psi = \varpi^2 h'(\psi)\ , \qquad E^2\left(E^2\psi\right) = 0 .$$

From the equations (8) there follows

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\right)\left(\varpi^2 h'(\psi)\right) = 0 ,$$

or

$$h'''(\psi)\,.\,\varpi^2\left(\psi_x^2 + \psi_\varpi^2\right) + h''(\psi)\left\{\varpi^2\left(\psi_{xx} + \psi_{\varpi\varpi} - \frac{1}{\varpi}\psi_\varpi\right) + 4\varpi\psi_\varpi\right\} = 0$$

i.e.,

$$(9)\qquad |\vec{q}|^2 = -\left\{\frac{h'(\psi)\,h''(\psi)}{h'''(\psi)} + \left(\frac{4}{\varpi^3}\psi_\varpi\right)\frac{h''(\psi)}{h'''(\psi)}\right\}$$

It follows from equation (9) that in all motions of the class under study, the isovels (i.e., the lines on which the magnitude of velocity is constant) cannot coincide with the streamlines except in the cases where the flow is purely radial or purely axial.

It is an interesting problem to examine the flow patterns common to the distinct classes of flows of inviscid and viscous liquids. Hamel[2] appears to have shown that the most general flow patterns common to steady, plane potential flows of a non-viscous, incompressible fluid and steady, plane flows of a viscous incompressible fluid are those corresponding to spiral flows. In the case of axisymmetric flows, if $\bar{\psi}$ and ψ are the stream functions of a potential motion of an inviscid liquid , and the motion of a viscous liquid of the type considered in this note, we have the equations

$$\bar{\psi} = f(\psi)\,, \quad E^2\bar{\psi} = 0\,, \quad \frac{E^2\psi}{\varpi^2} = h'(\psi)\,, \quad E^4\psi = 0\,.$$

Hence $E^2 f(\psi) = 0$ or $f''(\psi).\left(\psi_x^2 + \psi_\varpi^2\right) + f'(\psi).\left(\psi_{xx} + \psi_{\varpi\varpi} - \frac{1}{\varpi}\psi_\varpi\right) = 0$

i.e., $\varpi^2\,|\vec{q}|^2 = -\dfrac{f'(\psi)}{f''(\psi)}\,E^2\psi$

or $|\vec{q}|^2 = -\left\{f'(\psi)\,/\,f''(\psi)\right\}\,h'(\psi)\,.$

This means that the viscous flow is such that the isovels and the streamlines are coincident, and we have seen above that this is possible only in the trivial cases of purely radial or purely axial flows.

Hence we have the result:

The flow patterns common to steady axisymmetric potential flows of inviscid liquids and steady axisymmetric flows of viscous liquids of the present type are those corresponding to trivial flows only.

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THE AREA AND THE EQUATION OF THE GENERAL AEROFOIL SECTION

(To be published in the American Mathematical Monthly)

THE AREA AND THE EQUATION OF THE GENERAL AEROFOIL S[illegible]TION

Mancill and Betty Thomas[1] have determined the equation of a Joukowski aerofoil section by a simple elimination process. Mancill[2] has also given an expression for the area enclosed by the Joukowski profile using complex integration. The object of the present note is to obtain the equation of the general aerofoil section and also evaluate the area enclosed by the general aerofoil profile by the same methods.

The circle $|z - (ae^{i\beta} - c)| = a$, (c, β real) in the Z-plane is mapped onto a general profile by the transformation

(1) $$w = z + \frac{K}{z} + \frac{K_2}{z^2} + \cdots + \frac{K_n}{z^n}$$

This transformation has $n+1$ singular points at $z = -c, a_1, \ldots, a_n$ and the last n points are interior to the circle while the point $z = -c$ is on thecircumference. $K_1, K_2, \ldots, K_n$ are general complex constants and are expressible in terms of $a_1, a_2, \ldots, a_n$.

We have then

(2) $$(z + c - ae^{i\beta})(\bar{z} + c - ae^{-i\beta}) = a^2$$

(3) $$z^{n+1} - wz^n + Kz^{n-1} + K_2z^{n-2} + \cdots + K_{n-1}z + K_n = 0$$

(4) $$\bar{z}^{n+1} - \bar{w}\bar{z}^n + \bar{K}\bar{z}^{n-1} + \bar{K}_2\bar{z}^{n-2} + \cdots + \bar{K}_{n-1}\bar{z} + \bar{K}_n = 0$$

From (2) we have

(5) $$\bar{z} = (\bar{p}z - d)/(z - p)$$

where $p = ae^{i\beta} - c$, $d = p\bar{p} - a^2$.

Eliminating $\bar{z}$ from (4) and (5) we have

(6) $$(\bar{p}z - d)^{n+1} - \bar{w}(z-p)(\bar{p}z - d)^n + \bar{K}(z-p)^2(\bar{p}z - d)^{n-1} + \cdots + \bar{K}_n(z-p)^{n+1} = 0$$

The resultant of the two th degree polynomial equations (3) and (6) gives the equation of the general aerofoil section in isotropic coordinates $W = u + iv$, $\bar{W} = u - iv$ in the W-plane. For the choice $a_1 = c$, $a_2 = c'e^{i\theta}$, $a_3 = c'e^{-i\theta}$ with $n = 3$ we have the transformation

$$W = z + \frac{c^2 + c'^2 e^{2i\theta}}{z} - \frac{(c\,c' e^{i\theta})^2}{3z^3}$$

so that $K = c^2 + c'^2 e^{2i\theta}$, $K_2 = 0$, $K_3 = -\frac{1}{3}(c\,c'e^{i\theta})^2$.

The profile corresponding to this transformation is refferred to as the S-form profile[3] and its equation can be obtained as the resultant of the equations

$$z^4 - W z^3 + K z^2 + K_3 = 0,$$

$$(\bar{p}z - c)^4 - \bar{W}(z-p)(\bar{p}z-c)^3 + \bar{K}(z-p)^2(\bar{p}z-c)^2 + \bar{K}_3(z-p)^4 = 0.$$

The last equation is $A_0 z^4 + A_1 z^3 + A_2 z^2 + A_3 z + A_4 = 0$ where

$$A_0 = \bar{p}^4 - \bar{W}\bar{p}^3 + \bar{K}\bar{p}^2 + \bar{K}_3, \quad A_1 = -4\bar{p}^3 c + \bar{W}\bar{p}^2(p\bar{p}+3c) - 2\bar{K}\bar{p}(p\bar{p}+c) - 4\bar{K}_3 p,$$

$$A_2 = 6\bar{p}^2c^2 - 3\bar{W}\bar{p}c(p\bar{p}+c) + \bar{K}(p^2\bar{p}^2 + 4p\bar{p}c + c^2) + 6\bar{K}_3 p^2,$$

$$A_3 = -4\bar{p}c^3 + \bar{W}c^2(3p\bar{p}+c) - 2\bar{K}pc(p\bar{p}+c) - 4\bar{K}_3 p^3,$$

$$A_4 = c^4 - \bar{W}pc^3 + \bar{K}p^2c^2 + \bar{K}_3 p^4$$

The equation of the S-form profile in isotropic coordinates $(W, \bar{W})$ is therefore given by

$$\begin{vmatrix}
A_0 & A_1 & A_2 & A_3 & A_4 & 0 & 0 & 0 \\
0 & A_0 & A_1 & A_2 & A_3 & A_4 & 0 & 0 \\
0 & 0 & A_0 & A_1 & A_2 & A_3 & A_4 & 0 \\
0 & 0 & 0 & A_0 & A_1 & A_2 & A_3 & A_4 \\
1 & -W & K & 0 & K_3 & 0 & 0 & 0 \\
0 & 1 & -W & K & 0 & K_3 & 0 & 0 \\
0 & 0 & 1 & -W & K & 0 & K_3 & 0 \\
0 & 0 & 0 & 1 & -W & K & 0 & K_3
\end{vmatrix} = 0$$

The cartesian equation in the (u , v) plane is obtained by writing $w = u + iv$, $\bar{w} = u - iv$ and it is of the 8th degree in general. When $p = 0$ this equation is of the 6th degree. To write the equation in full would be very cumbersome.

AREA

Mancill[2] has given the formula

$$A(C) = \frac{1}{2i} \int_C \bar{z}\, dz$$

for the area of a closed curve. For the general aerofoil (L) we have then

$$A(L) = \frac{1}{2i} \int_{(L)} \bar{w}\, dw = \frac{1}{2i} \int \left(\bar{z} + \frac{\bar{K}}{\bar{z}} + \frac{\bar{K}_2}{\bar{z}^2} + \cdots + \frac{\bar{K}_n}{\bar{z}^n}\right)\left(1 - \frac{K}{z^2} - \frac{2K_2}{z^3} - \cdots - \frac{nK_n}{z^{n+1}}\right) dz$$

where C_0 is the mapping circle.,viz.,

To evaluate the integral we note the følloving results:

$$\frac{1}{2i} \int_{C_0} \bar{z}\, dz = \pi (p\bar{p} - c) = \pi a^2; \quad \int_{C_0} \frac{dz}{\bar{z}^m} = 0 \quad (m = 1, 2, \ldots, n);$$

$$\int_{C_0} \frac{\bar{z}}{z^m}\, dz = 0 \quad (m = 1, 2, \ldots, n+1); \quad \frac{1}{2i} \int \frac{dz}{\bar{z}^m z^l} = \frac{\pi}{(l-1)!} \left\{ \left(\frac{d}{dz}\right)^{l-1} \left(\frac{z - p}{\bar{p}z - c}\right)^m \right\}_{z=0}.$$

$$A(L) = \pi \left\{ a^2 - K \sum_{n=1}^{n} \bar{K}_n \left[\frac{d}{dz} \left(\frac{z-p}{\bar{p}z - c}\right)^n \right]_{z=0} - \frac{2K_2}{2!} \sum_{n=1}^{n} \bar{K}_n \left[\left(\frac{d}{dz}\right)^2 \left(\frac{z-p}{\bar{p}z - c}\right)^n \right]_{z=0} \right.$$

$$\left. - \cdots\cdots - \frac{nK_n}{n!} \sum_{n=1}^{n} \bar{K}_n \left[\left(\frac{d}{dz}\right)^n \left(\frac{z-p}{\bar{p}z - c}\right)^n \right]_{z=0} \right\}$$

where in each summation on the right $\bar{K}_n = K$ when $n = 1$.

For the 3-form profile we have

$$A(L) = \pi \left\{ a^2 - (c^2 + c'^2 e^{2i\theta}) \sum_{n=1}^{3} \bar{K}_n \left[\frac{d}{dz} \left(\frac{z-p}{\bar{p}z - c}\right)^n \right]_{z=0} + \frac{(cc'e^{i\theta})^2}{6} \sum_{n=1}^{3} \bar{K}_n \left[\left(\frac{d}{dz}\right)^3 \left(\frac{z-p}{\bar{p}z - c}\right)^n \right. \right.$$

$$= \pi a^2 \left\{ 1 - \frac{|c^2 + c'^2 e^{2i\theta}|^2}{(p\bar{p} - a^2)^2} + \frac{(c^2 + c'^2 e^{2i\theta})(cc'e^{-i\theta})^2 p^2}{(p\bar{p} - a^2)^4} + \frac{(c^2 + c'^2 e^{2i\theta})(cc'e^{i\theta})^2 p^2}{(p\bar{p} - a^2)^4} \right.$$

$$\left. - \frac{1}{3} |cc'e^{i\theta}|^4 \left(3 p^2\bar{p}^2 + 6 p\bar{p} a^2 + a^4\right) \Big/ (p\bar{p} - a^2)^6 \right\}.$$

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VIBRATIONS OF COMPOSITE STRINGS

(Joint work with Dr.B.S.RamaKrishna)

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VIBRATIONS OF COMPOSITE STRINGS

Vibrations of composite bodies are sometimes encountered in practice, such as in some types of Indian musical drums, and are often difficult to solve exactly under the particular boundary conditions obtained. The simplest of these, the vibrations of a composite string leads to a typical, though not novel, boundary value problem, and lends itself to analysis by elementary methods.

We consider here the vibrations of a composite string consisting of two ideal strings of linear densities ϵ_1 and ϵ_2 and total length unity, (which involves no loss of generality) joined together at h $(0<h<1)$ and held fixed at its ends under uniform tension T. The end in view is to determine the totality of eigen-values and eigen-functions of the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \begin{cases} c_1^2 \dfrac{\partial^2 u}{\partial x^2} & 0 \leq x \leq h \\ c_2^2 \dfrac{\partial^2 u}{\partial x^2} & h \leq x \leq 1 \end{cases} \tag{1}$$

where

$$c_1^2 = T/\epsilon_1 \qquad \text{and} \qquad c_2^2 = T/\epsilon_2$$

$$u(0,t) = u(1,t) = 0 \tag{2}$$

$$u(h-0,t) = u(h+0,t) \tag{3}$$

$$\frac{\partial u(h-0,t)}{\partial x} = \frac{\partial u(h+0,t)}{\partial x} \tag{4}$$

for arbitrary values of h and the ratio $r = \sqrt{\epsilon_1/\epsilon_2}$. The boundary conditions (3) and (4) simply assert the continuity of the vertical displacement and the slope of the string across the join of the parts.

The usual assumptions of harmonic vibrations

$$u(x,t) = \begin{cases} Y_1(x)\, e^{i\omega t} & 0 \leq x \leq h \\ Y_2(x)\, e^{i\omega t} & h \leq x \leq 1 \end{cases} \tag{5}$$

leads to the equations

$$\begin{aligned} \frac{d^2 Y_1}{dx^2} + \lambda_1^2 Y_1 &= 0 \qquad 0 \leq x \leq h \\ \frac{d^2 Y_2}{dx^2} + \lambda_2^2 Y_2 &= 0 \qquad h \leq x \leq 1 \end{aligned} \tag{6}$$

where

$$\lambda_1^2 = \omega^2/c_1^2 \qquad \text{and} \qquad \lambda_2^2 = \omega^2/c_2^2 \tag{7}$$

are the eigen-values parameters which determine the natural frequencies of vibration ω. In view of the boundary conditions (2), we take for the solution of (6)

$$\begin{aligned} Y_1 &= A_1 \sin \lambda_1 x & 0 \le x \le h \\ Y_2 &= A_2 \sin \lambda_2 (1-x) & h \le x \le 1 \end{aligned} \tag{8}$$

The boundary conditions (4) lead to the characteristic equation

$$\lambda_2 \cos \lambda_2 (1-h) \sin \lambda_1 h + \lambda_1 \cos \lambda_1 h \sin \lambda_2 (1-h) = 0 \tag{9}$$

Replacing λ_1, and λ_2 from (7) in (9) and using the relation $c_2/c_1 = \sqrt{\epsilon_1/\epsilon_2} = r$, we obtain the frequency equation directly in terms of ω.

$$\cos \frac{\omega(1-h)}{r c_1} \sin \frac{\omega h}{c_1} + r \cos \frac{\omega h}{c_1} \sin \frac{\omega(1-h)}{r c_1} = 0 \tag{10}$$

Writing $\omega h/c_1 = x$ and $(1-h)/rh = \alpha$, we obtain

$$\cos \alpha x \sin x + r \cos x \sin \alpha x = 0, \tag{11}$$

a form more suitable for numerical computation.

The roots of the transcendental equation (11) lead to a discrete spectrum of eigen-values x_K which in general depend on both r and α. Among the eigen-values x_K for rational values α, a paticular sub-sequence of eigen-values x'_K, which contains the fundamental, deserves special mention. It will be shown below that when α is rational, there exist eigen-values x'_K for which both the terms in equation (11) are simultaneously zero. The corresponding subset of eigen-functions Y' have the simple physical meaning that the join of the two parts, $x = h$ is either a node or an anti-node for each of these modes. Conversely, the join point of the composite string can be a node or anti-node only when α is rational.

This special subsequence of eigen-values and eigen-functions will be first obtained. These are determined by both the terms in equation (11) vanishing simultaneously, i.e.,

$$\cos \alpha x \sin x = 0 ,$$
$$\cos x \sin \alpha x = 0 .$$

Since the sine and cosine temms can never simultaneously vanish, only two possibilities exist,

$$\sin x = 0 , \quad \sin \alpha x = 0 \tag{12}$$

or

$$\cos x = 0 , \quad \cos \alpha x = 0 \tag{13}$$

It is readily seen that for $\sin x$ and $\sin \alpha x$ to have coincident zeros, a prerequisite is that α must be rational:

$$\alpha = \frac{m}{n} \tag{14}$$

m and n being in their lowest form.

However, if both $\cos x$ and $\cos \alpha x$ are to have coincident zeros, α must be further restricted to a rational number of the form

$$\alpha = (2m+1) / (2n+1) \tag{15}$$

We distinguish two cases; the first in which α is of the form

$$\alpha = m/n \tag{14'}$$

where both m. and n are not odd integers, and the second in which α is of the form

$$\alpha = (2m+1) / (2n+1) \tag{15'}$$

In the first case, only the sine terms can vanish and the coincident roots of $\sin x$ and $\sin \alpha x$ lead to the eigen-values

$$x_K' = K n \pi \qquad (K = 1, 2, \cdots \quad) \tag{16}$$

In the second case where is of the form (15') it is possible for both pairs of equations (12) and (13) to be true and we obtain the set of eigen-values

$$x_K = K \; (2n+1) \frac{\pi}{2} \tag{17}$$

even values of k giving the coincident roots of the pair of equations (12) and odd values of k giving the coincident roots of the pair of equations (13).

We now write down the corresponding eigen-functions. We note that only the ratio of the amplitudes is determined for each mode, this being obtained from one of the equations(3) and (4). The eigen-functions corresponding to (16) are given by

$$Y_1^{(K)}(x) = A \sin \frac{K n \pi}{h} x \quad , \quad 0 \leq x \leq h$$
$$Y_2^{(K)}(x) = (-1)^{K(n-m)+1} A n \sin \frac{K n \pi}{n h}(1-x) \quad , \quad h \leq x \leq 1 \qquad (18)$$

Similarly the eigen-functions corresponding to even values of k in (17) are given by

$$Y_1^{(K)}(x) = A \sin \frac{K(2n+1)\pi}{2h} x \qquad 0 \leq x \leq h$$
$$Y_2^{(K)}(x) = -A n \sin \frac{K(2n+1)\pi}{2h}(1-x) \qquad h \leq x \leq 1 \qquad (K = 2, 4 \ldots) \quad (19)$$

while the eigen-functions corresponding to odd values of k in (17) are

$$Y_1^{(K)}(x) = A \sin \frac{K(2n+1)\pi}{2h} x \quad , \quad 0 \leq x \leq h$$
$$Y_2^{K}(x) = (-1)^{K(n-m)} A \sin \frac{K(2n+1)\pi}{2n h}(1-x), \quad h \leq x \leq 1 \qquad (K = 1, 3, \cdots) \quad (20)$$

It will be noted that the point $x = h$ is a node for each of the eigen-functions (18) and (19) and an anti-node for the eigen-functions (20). The fundamental mode of vibration is thus obtained by putting $k=1$ in (20).

The eigen-values x_K' given by equations (16) and (17) and the corresponding eigen-functions given by equations (18)-(20) are somewhat special in the sense that while the x_K' are the roots of (11), they make the individual terms vanish simultaneously. We have yet to obtain the eigen-values x_K which are the roots of (11), but for which the individual terms do not vanish. These are therefore the roots of

$$\tan x + r \tan \alpha x = 0 \qquad (21)$$

The eigen-values x_K are in general functions of both r $(0 < r < \infty)$ and α $(0 < \alpha < \infty)$ and have to be determined graphically except in very special cases. The graphical work required may be considerably reduced by plotting $y = \tan x / \tan \alpha x$ with α as a parameter, when the roots $x_K(\alpha)$ corresponding to any r may be read off directly as the abscissae of the intersections of these curves with y r. The actual computations therefore need to take into account only one of the two variable parameters, viz., α . Further, the range of r needed in the graphical work may be restricted to $0 < r < 1$; for, as will be shown below, the eigen-values for any r ($1 < r < \infty$) and α ($0 < \alpha < \infty$) are direct multiples by the factor α of the eigen-values corresponding to $\frac{1}{\pi}$ ($0 < \frac{1}{\pi} < 1$) and $1/\alpha$. To prove this, let

$$f(\pi, \alpha; x) \equiv \tan x + \pi \tan \alpha x \tag{22}$$

then we note that

$$\pi f(\tfrac{1}{\pi}, \tfrac{1}{\alpha}; x) \equiv f(\pi, \alpha; \tfrac{x}{\alpha})$$

which proves the above assertion.

Physically, the two composite strings with r and , and 1/r and $1/\alpha$ are identical (for any fixed r and α) except for the interchange of the component parts, and equation (22) is merely a statement of the obvious fact that the frequency of vibration ω_k remains the same when the two parts are interchanged. However, a knowledge of this fact enables us to confine ourselves to plotting the curves in the strip of width $-1 < \pi < 0$. It may also be noted in passing that if α is a rational number m/n, the graph can be confined to $0 < x < n\pi$ as $f(r, \alpha; x)$ is then periodic with period $n\pi$.

Fig.1 shows the various branches of $\tan x / \tan \alpha x$ in the range for 2,4,6,8.

As a particularly simple illustration of the above analysis, we consider a composite string made of two parts of nearly equal linear densities and of equal length. Accordingly, let

$$\epsilon_1 = \epsilon \; , \qquad \epsilon_2 = \epsilon(1+\delta) \qquad |\delta| << 1$$

where may be positive or negative. Hence

$$\pi \simeq 1 - \frac{\delta}{2} \; , \qquad \alpha = \pi^{-1} \simeq 1 + \frac{\delta}{2}$$

and we obtain for the eigen-value equation

$$\cos\left(1+\frac{\delta}{2}\right)x \sin x + \left(1-\frac{\delta}{2}\right)\cos x \sin\left(1+\frac{\delta}{2}\right)x = 0$$

Neglecting in comparison with unity in the coefficient of the second term we obtain for a first approximation

$$\sin\left(2+\frac{\delta}{2}\right)x = 0$$

which leads to the eigen-values

$$x_K = K\pi\left(1-\frac{\delta}{4}\right)/2 \qquad (K=1,2,\dots)$$

The corresponding frequencies of vibrations

$$\omega_K \simeq 2c\,x_K = K\pi c\left(1-\frac{\delta}{4}\right)$$

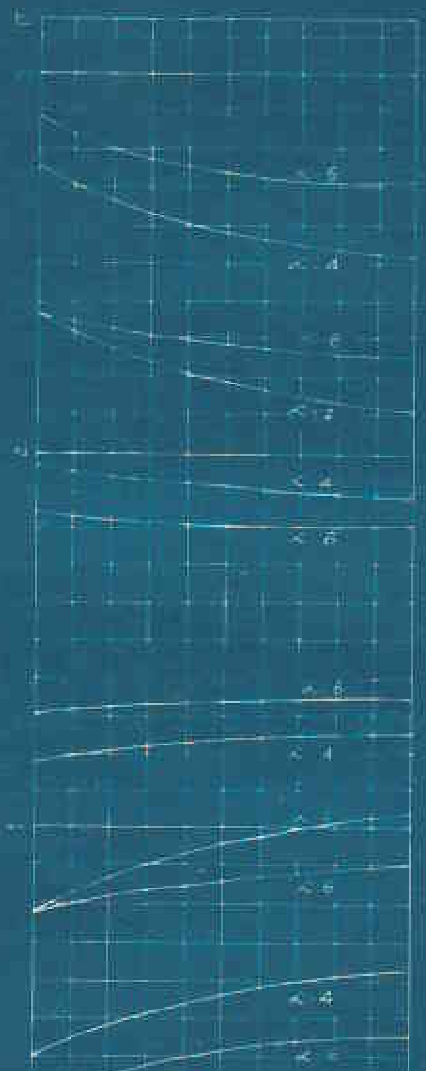
are lower than those of a string of uniform density ϵ if δ is positive, and conversely.

Finally, we write down the eigen-functions corresponding to the roots x_K of equation (21).

$$Y_1^{(K)} = A \sin\frac{x_K}{h}x, \qquad 0 \leq x \leq h$$

$$Y_2^{(K)} = A\frac{\sin x_K}{\sin\alpha x_K}\sin x_K\left(\frac{1-x}{\alpha h}\right) \qquad h \leq x \leq 1$$

The orthogonality of these functions with the density as the weight, may be verified in the usual manner. In contrast to these functions, when α is rational, the eigen-functions (18)-(20) are orthogonal in each of the ranges ($0 \leq x \leq h$) and ($h \leq x \leq 1$) separately.



A PROBLEM IN HEAT CONDUCTION AND THE PRODUCT OF THREE ERROR FUNCTIONS

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A PROBLEM IN HEAT CONDUCTION AND THE PRODUCT OF THREE ERROR FUNCTIONS

Rainville[1] has obtained an expension for the product of two error functions by solving a two-dimensional problem in heat conduction. He obtains the temperature distribution in a quadrantal infinite solid initially at a constant temperature (which may be taken as unity) and whose two faces are maintained at zero temperature in two different forms and their identification provides the relation

$$(1)\quad \operatorname{erf}(r\cos\phi)\operatorname{erf}(r\sin\phi) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(4n+2)\phi \cdot r^{4n+2}\,{}_1F_1(2n+1;\,4n+3;\,-r^2)}{(2n+1)\,2^{4n+1}\left(\frac{3}{2}\right)_{2n}}$$

where the error function is defined by

$$(2)\quad \operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\alpha^2}\,d\alpha = \frac{2x}{\sqrt{\pi}}\,{}_1F_1\left(\tfrac{1}{2};\,\tfrac{3}{2};\,-x^2\right)$$

and ${}_1F_1(a;b;x)$ is the confluent hypergeometric function defined by

$$(3)\quad {}_1F_1(a;\,b;\,x) = \sum_{n=0}^{\infty}\frac{(a)_n\,x^n}{n!\,(b)_n}$$

and $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma a}$ $(n = 0, 1, 2, \cdots)$. b is neither zero nor a negative integer.

Presently we study a similar problem in three dimensions. An octantal solid bounded by the quadrants of planes $x=0$, $y=0$, and $z=0$ and otherwise extending to infinity such that the solid occupies the region $0 < x, y, z$ with no restriction on the right side is kept initially at a constant temperature (which is taken as unity) and the faces $x=0$, $y=0$, and $z=0$ are kept at zero temperature for $t > 0$. We determine the temperature function in the solid in two ways in a manner entirely similar to Rainville's. We may then deduce an expansion for the product of three error functions by identifying the two forms of solution of one and the same physical problem. However it must be noted that the last step viz., of equating the two forms of solution is valid only if the physical problem is shown to have a unique solution. No attempt is made here in this direction.

The temperature function $u(x, y, z, t)$ in the octantal solid is a solution of the boundary value problem stated by the following equations.

(4) $$\frac{\partial u}{\partial t} = h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad t>0;\; x>0,\; y>0,\; z>0$$

(5) As $t \to 0^+$, $u \to 1$ $\qquad x>0,\; y>0,\; z>0.$

(6) As $x \to 0^+$, $u \to 0$ $\qquad t>0;\; y>0,\; z>0.$

(7) As $y \to 0^+$, $u \to 0$ $\qquad t>0;\; z>0,\; x>0.$

(8) As $z \to 0^+$, $u \to 0$ $\qquad t>0;\; x>0,\; y>0.$

in (4) is the thermal diffusivity and is assumed constant.

It is easily verified that the problem described by the above equations (4)-(8) has the solution

(9) $$u = \operatorname{erf}\left(\frac{x}{2h\sqrt{t}}\right)\operatorname{erf}\left(\frac{y}{2h\sqrt{t}}\right)\operatorname{erf}\left(\frac{z}{2h\sqrt{t}}\right)$$

This is ofcourse a classical result.

Let us now employ spherical polar coordinates ρ, θ, ϕ for stating the same physical problem. ρ, θ, ϕ are related to the cartesian coordinates through the usual equations $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\theta$. The boundary value problem is now described as follows.

(10) $$\frac{\partial u}{\partial t} = h^2\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial u}{\partial \theta} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right)$$
$$t>0;\; \rho>0,\; 0<\theta<\tfrac{\pi}{2},\; 0<\phi<$$

(11) As $t \to 0^+$, $u \to 1$ $\qquad \rho>0,\; 0<\theta<\frac{\pi}{2},\; 0<\phi<\frac{\pi}{2}.$

(12) As $\phi \to \left(\frac{\pi}{2}\right)^-$, $u \to 0$ $\qquad t>0;\; \rho>0,\; 0<\theta<\frac{\pi}{2}.$

(13) As $\phi \to 0^+$, $u \to 0$ $\qquad t>0;\; \rho>0,\; 0<\theta<\frac{\pi}{2}.$

(14) As $\theta \to \left(\frac{\pi}{2}\right)^-$, $u \to 0$ $\qquad t>0;\; \rho>0,\; 0<\phi<\frac{\pi}{2}.$

To solve the problem stated in the equations (10)-(14), following Rainville we seek certain types of solutions of the differential equation (10) and then take care of the subsidiary conditions (11)-(14). We choose possible solutions of (10) depending only on three arguments in the form

(15) $$u = \psi\left(-\frac{\rho^2}{4h^2t}, \theta, \phi\right) \equiv \psi(v, \theta, \phi)$$

with $v = -\dfrac{\rho^2}{4h^2t}$. Equation (10) is then recast into

(16) $$v^2 \frac{\partial^2\psi}{\partial v^2} + v\left(\frac{3}{2}-v\right)\frac{\partial\psi}{\partial v} + \frac{1}{4}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{4}\frac{\partial\psi}{\partial\theta} + \frac{\operatorname{cosec}^2\theta}{4}\frac{\partial^2\psi}{\partial\phi^2} = 0$$
$$(v < 0;\quad 0 < \theta < \tfrac{\pi}{2},\quad 0 < \phi < \tfrac{\pi}{2})$$

The conditions (11)-(14) now become

(17) As $v \to -\infty$, $\psi \to 1$ $\qquad 0 < \theta < \frac{\pi}{2}$; $0 < \theta < \frac{\pi}{2}$.

(18) As $\phi \to \left(\frac{\pi}{2}\right)^-$, $\psi \to 0$ $\qquad v < 0$; $0 < \theta < \frac{\pi}{2}$.

(19) As $\phi \to 0^+$, $\psi \to 0$ $\qquad v < 0$; $0 < \theta < \frac{\pi}{2}$.

(20) As $\theta \to \left(\frac{\pi}{2}\right)^-$, $\psi \to 0$ $\qquad v < 0$; $0 < \phi < \frac{\pi}{2}$.

The boundary value problem (16)-(20) may now be solved by the method of separation of variables. Let $\psi = V(v)\,.\,\Theta(\theta)\cdot\Phi(\phi)$. (16) then takes the form

(21) $$\frac{1}{V}\left(v^2\frac{d^2V}{dv^2} + v\left(\frac{3}{2}-v\right)\frac{dV}{dv}\right) + \frac{1}{4\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta}\right) + \frac{\operatorname{cosec}^2\theta}{4\Phi}\frac{d^2\Phi}{d\phi^2} = 0.$$

which can be changed into

(22) $$\frac{\sin^2\theta}{V}\left(v^2\frac{d^2V}{dv^2} + v\left(\frac{3}{2}-v\right)\frac{dV}{dv}\right) + \frac{\sin^2\theta}{4\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta}\right) = -\frac{1}{4\Phi}\frac{d^2\Phi}{d\phi^2} = \alpha^2$$

The equation $\dfrac{d^2\Phi}{d\phi^2} + 4\alpha^2\phi = 0$ has the general integral $A\cos 2\alpha\phi + B\sin 2\alpha\phi$ and from the boundary conditions (18) and (19) it is clear that

(23) $$\Phi = \sin 2m\phi \qquad (m = 1, 2, \cdots)$$

It follows now from (22) that

(24) $$\frac{1}{V}\left(v^2\frac{d^2V}{dv^2} + v\left(\frac{3}{2}-v\right)\frac{dV}{dv}\right) = -\frac{1}{4\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = \beta^2$$

The equation

(25) $\frac{d^2 \Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + \left(4\beta^2 - \frac{4m^2}{\sin^2\theta}\right)\Theta = 0$

can be solved in terms of the Legendre's functions and to satisfy (20) β^2 is to be quantized according to the equation $4\beta^2 = (2n+1)(2n+2)$; then the relevent solution of (25) is

(26) $\Theta = P^{2m}_{2n+1}(\cos\theta) \qquad n, m = 1, 2, 3, \cdots \quad (n \geq m)$

$V(v)$ is then a solution of

(27) $v^2 \frac{d^2 V}{dv^2} + v\left(\frac{3}{2} - v\right)\frac{dV}{dv} - \frac{(n+1)(2n+1)}{2} V = 0 .$

We may solve this equation by the method of integration in series and find that $V = c_1 V_1 + c_2 V_2$ where

$V_1 = v^{n+\frac{1}{2}} {}_1F_1(n+\frac{1}{2}; 2n+\frac{5}{2}; v)$ and $V_2 = v^{-(n+1)} {}_1F_1(-n-1; -2n-\frac{1}{2}; v)$.

V_2 is a polynomial in $\frac{1}{v}$ of degree $n+1$ and is to be discarded by us as ψ is finite when $v \to 0$ for any fixed t.

We may then take the solution $\psi(v, \theta, \phi)$ in the form

(28) $\psi = \sum_{n=1}^{\infty}\sum_{m=1}^{n} A_{nm} v^{n+\frac{1}{2}} {}_1F_1(n+\frac{1}{2}; 2n+\frac{5}{2}; v) P^{2m}_{2n+1}(\cos\theta) \sin 2m\phi$

All the conditions with the exception of (17) are already satisfied. To take care of this we note here that[2]

(29) $${}_1F_1(a; c; x) = \frac{\Gamma c}{\Gamma c-a}(-x)^{-a}\left(1 + O|x|^{-1}\right)$$

as Re $x \to -\infty$. Then we need to have

(30) $$1 = \sum_{n=1}^{\infty}\sum_{m=1}^{n} A_{nm} \frac{\Gamma 2n+\frac{5}{2}}{\Gamma n+2} e^{i\pi(n+\frac{1}{2})} P^{2m}_{2n+1}(\cos\theta) \sin 2m\phi .$$

$0 < \theta < \frac{\pi}{2}, \quad 0 < \phi < \frac{\pi}{2}$

Product of the two sides with $\sin 2p\phi$ on integration with respect to ϕ leads to

$$\int_0^{\pi/2} \sin 2p\phi \, d\phi = \frac{\pi}{4}\sum_{n=1}^{\infty} A_{np} \frac{\Gamma 2n+\frac{5}{2}}{\Gamma n+2} e^{i\pi(n+\frac{1}{2})} P^{2p}_{2n+1}(\cos\theta)$$

$(p = 1, 2, \cdots, n) \quad 0 < \theta < \frac{\pi}{2}$

As the integral in the left member $= 0$ for even values of p we take

$$(30)\qquad A_{np} = 0 \qquad (p = 2, 4, 6, \cdots \qquad)$$

and

$$(31)\qquad \frac{\pi}{4}\sum_{n=1}^{\infty} A_{np}\,\frac{\Gamma(2n+\frac{5}{2})}{\Gamma(n+2)}\,e^{i\pi(n+\frac{1}{2})}\,P_{2n+1}^{2p}(\cos\theta) = \frac{1}{p} \qquad (p = 1, 3, \cdots \leq n).$$

Product of the two members of this equation with $P_{2K+1}^{2p}(\cos\theta)\sin\theta$ and integration with respect to θ over $(0, \frac{\pi}{2})$ leads to

$$(32)\qquad \frac{\pi}{4}\,A_{Kp}\,\frac{\Gamma(2K+\frac{5}{2})}{\Gamma(K+2)}\,e^{i\pi(K+\frac{1}{2})}\,\frac{1}{4K+3}\,\frac{\Gamma(2K+2p+2)}{\Gamma(2K-2p+2)} = \frac{1}{p}\int_0^{\pi/2} P_{2K+1}^{2p}(\cos\theta)\sin\theta\,d\theta$$

As a closed form of expression is not known for the last integral we write the polynomial form of $P_{2K+1}^{2p}(x)$ and integrate term by term, deriving then

$$(33)\qquad A_{Kp} = \left(\frac{2}{\sqrt{\pi}}\right)^3 2^{2K}\,e^{-i\pi(K+\frac{1}{2})}\,\frac{\Gamma(p)\,\Gamma(K-p+1)}{\Gamma(2K+2p+2)}\sum_{\eta=0}^{K-p}(-1)^{\eta}\binom{K+1}{\eta}\frac{(2K-2p+1)(2K-2p-1)\cdots(2K-2p-\cdots}{(4K+1)(4K-1)\cdots(4K-2\eta\cdots}$$

$$(K = 1, 2, \cdots;\quad p = 1, 3, \cdots \leq K)$$

From (28) there follows

$$(34)\qquad \psi(v,\theta,\phi) = \left(\frac{2}{\sqrt{\pi}}\right)^3\sum_{n=1}^{\infty}{}'\sum_{m}^{1,3,\cdots\leq n} 2^{2n}\,e^{-i\pi(n+\frac{1}{2})}\left(\frac{\Gamma(m)\,\Gamma(n-m+1)}{\Gamma(2n+2m+2)}\right)$$

$$\sum_{\eta=0}^{n-m}{}'(-1)^{\eta}\binom{n+1}{\eta}\frac{(2n-2m+1)(2n-2m-1)\cdots(2n-2m-2\eta+3)}{(4n+1)(4n-1)\cdots(4n-2\eta+3)}\cdot v^{n+\frac{1}{2}}\,{}_1F_1(n+\tfrac{1}{2};\,2n+\tfrac{5}{2};\,v\cdots$$

$$\cdot P_{2n+1}^{2m}(\cos\theta)\sin 2m\phi$$

$$(v < 0,\ 0 < \theta, \phi < \cdots$$

Writing this solution in terms of ρ, θ, ϕ, t we have

$$(35)\qquad \psi\left(-\frac{\rho^2}{4h^2t}, \theta, \phi\right) = u(\rho, \theta, \phi, t)$$

$$= \left(\frac{2}{\sqrt{\pi}}\right)^3\sum_{n=1}^{\infty}{}' 2^{2n}\left(\frac{\rho^2}{4h^2t}\right)^{n+\frac{1}{2}}{}_1F_1\left(n+\tfrac{1}{2};\,2n+\tfrac{5}{2};\,-\frac{\rho^2}{4h^2t}\right)\sum_{m}^{1,3,\cdots\leq n}{}'\left(\frac{\Gamma(m)\,\Gamma(n-m+\cdots}{\Gamma(2n+2m+\cdots}\right.$$

$$\cdot P_{2n+1}^{2m}(\cos\theta)\sin 2m\phi\sum_{\eta=0}^{n-m}{}'(-1)^{\eta}\binom{n+1}{\eta}\frac{(2n-2m+1)(2n-2m-1)\cdots(2n-2\eta\cdots}{(4n+1)(4n-1)\cdots(4n-2\eta+3\cdots}$$

$$t > 0;\quad \rho > 0,\quad 0 < \phi < \frac{\pi}{2},\quad 0 < \theta < \frac{\pi}{2}$$

(9) and (35) are two forms of solution of one and the same boundary value problem. Before identifying them we may effect the transformation

$$x = 2h\sqrt{t}\, r \sin\theta \cos\phi, \quad y = 2h\sqrt{t}\, r \sin\theta \sin\phi, \quad z = 2h\sqrt{t}\, r \cos\theta$$

which means $\rho^2 = x^2+y^2+z^2 = 4h^2 t r^2$. Equating the expressions of the equations (9) and (35) we obtain the expansion

$$(36)\quad \mathrm{erf}(r\sin\theta\cos\phi)\,\mathrm{erf}(r\sin\theta\sin\phi)\,\mathrm{erf}(r\cos\theta) =$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^3 \sum_{n=1}^{\infty} 2^{2n} r^{2n+1}\, {}_1F_1\left(n+\tfrac{1}{2};\, 2n+\tfrac{5}{2};\, -r^2\right) \sum_{m}^{1,3,\cdots\leq n} B_{nm} \frac{m!\,(n-m)!}{(2n+2m+2)!} P_{2n+1}^{2m}(\cos\theta)\sin 2m \qquad r>0,\ 0\leq\theta,\phi$$

where $$B_{nm} = \sum_{r=0}^{n-m} (-1)^r \binom{n+1}{r} \frac{(2n-2m+1)(2n-2m-1)\cdots(2n-2m-2r+3)}{(4n+1)(4n-1)\cdots(4n-2r+3)}$$

As mentioned above the validity of (36) is open to question until it is rigorously shown that that the physical problem has a unique solution.

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THE TORSIONAL RIGIDITY OF A BAR WITH SECTORIAL CROSS-SECTION

THE TORSIONAL RIGIDITY OF A BAR WITH SECTORIAL CROSS-SECTION

The analytical solution of the problem of torsion of a bar with sectorial cross-section is a classical result due to Stokes. However the torsional rigidity does not seem to have been expressed in a convenient form for the sectorial bar except in the case of the full and semi-circular bars. We obtain in the present note, an expression for the torsional rigidity of a bar with a sector of angle as cross-section, in terms of the Psi- functions and their derivatives, for which prepared tables of values are available.

The stress function for such a bar is[1]

$$(1)\quad \phi = \frac{G\theta}{2}\left\{-r^2\left(1-\frac{\cos 2\psi}{\cos\alpha}\right) + a^2\sum_{n=1,3,\dots}^{\infty} A_n\left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}}\cos\frac{n\pi\psi}{\alpha}\right\}$$

with $A_n = \frac{16\alpha^2}{\pi^3}(-1)^{\frac{n+1}{2}} \Big/ n\left(n+\frac{2\alpha}{\pi}\right)\left(n-\frac{2\alpha}{\pi}\right)$.

The torsional rigidity for any cylindrical bar is given by $M_t = 2\iint \phi\, r\, dr\, d\psi$

Hence

$$\frac{M_t}{G\theta} = \int_{-\alpha/2}^{\alpha/2} d\psi \int_0^a \left\{-r^2\left(1-\frac{\cos 2\psi}{\cos\alpha}\right) + a^2\sum_{n=1,3,\dots}^{\infty} A_n\left(\frac{r}{a}\right)^{n\pi/\alpha}\cos\frac{n\pi\psi}{\alpha}\right\} r\, dr .$$

On performing the integration, we have

$$\frac{M_t}{G\theta} = \frac{a^4}{4}\left(\tan\alpha-\alpha\right) - \frac{32a^4\alpha^4}{\pi^5}\sum_{1,3,\dots}^{\infty} 1\Big/ n^2\left(n-\frac{2\alpha}{\pi}\right)\left(n+\frac{2\alpha}{\pi}\right)^2 .$$

To evaluate the sum of the series on the right we write

$$\sum_{n=1,3,\dots}^{\infty} 1\Big/ n^2\left(n-\frac{2\alpha}{\pi}\right)\left(n+\frac{2\alpha}{\pi}\right)^2 = \frac{\pi^3}{16\alpha^3}\sum_{1,3,\dots}^{\infty}\left(n^2-\frac{4\alpha^2}{\pi^2}\right)^{-1} + \frac{\pi^4}{16\alpha^4}\sum_{1,3,\dots}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{2\alpha}{\pi}}\right)$$

$$-\frac{\pi^3}{8\alpha^3}\sum_{1,3,\dots}^{\infty} n^{-2} - \frac{\pi^3}{16\alpha^3}\sum_{1,3,\dots}^{\infty}\left(n+\frac{2\alpha}{\pi}\right)^{-2}$$

$$= \frac{\pi^5}{16\alpha^3}\sum_{1,3,\dots}^{\infty}\left(\pi^2 n^2-4\alpha^2\right)^{-1} + \frac{\pi^4}{16\alpha^4}\left\{\sum_1^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{2\alpha}{\pi}}\right) - \frac{1}{2}\sum_1^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{\alpha}{\pi}}\right)\right\}$$

$$-\left(\pi^5/64\alpha^3\right) - \frac{\pi^3}{16\alpha^3}\left\{\sum_1^{\infty}\left(n+\frac{2\alpha}{\pi}\right)^{-2} - \frac{1}{4}\sum_1^{\infty}\left(n+\frac{\alpha}{\pi}\right)^{-2}\right\}$$

Using the partial fraction expansions of $\tan\alpha$, $\psi(z) = \frac{d}{dz}\log\Gamma z$ and $\psi'(z)$ the above series reduces to

$$\frac{\pi^5}{16\alpha^3}\cdot\frac{\tan\alpha}{8\alpha} + \frac{\pi^4}{16\alpha^4}\left\{\psi\left(1+\frac{2\alpha}{\pi}\right) - \frac{1}{2}\psi\left(1+\frac{\alpha}{\pi}\right) - \frac{1}{2}\psi(1)\right\} - \frac{\pi^5}{64\alpha^3}$$

$$-\frac{\pi^3}{16\alpha^3}\left\{\psi'\left(\frac{2\alpha}{\pi}\right) - \frac{1}{4}\psi'\left(\frac{\alpha}{\pi}\right)\right\}.$$

We then have

$$\frac{M_t}{G\theta} = \frac{a^4\alpha}{4} - \frac{2a^4}{\pi}\left\{\psi\left(1+\frac{2\alpha}{\pi}\right) - \frac{1}{2}\psi\left(1+\frac{\alpha}{\pi}\right) - \frac{1}{2}\psi(1)\right\}$$

$$+ \frac{2a^4\alpha}{\pi^2}\left\{\psi'\left(\frac{2\alpha}{\pi}\right) - \frac{1}{4}\psi'\left(\frac{\alpha}{\pi}\right)\right\}$$

From the relations[2] $\psi(mz) = m^{-1}\sum_{n=0}^{m-1}\psi\left(z+\frac{n}{m}\right) + \log m$ and $\psi'(mz) = m^{-2}\sum_{n=0}^{m-1}\psi'\left(z+\frac{n}{m}\right)$

we can also write

$$\frac{M_t}{G\theta} = \frac{a^4\alpha}{4} - \frac{2a^4}{\pi}\left\{\frac{1}{2}\psi\left(\frac{1}{2}+\frac{\alpha}{\pi}\right) + \log 2 - \frac{1}{2}\psi(1)\right\} + \frac{a^4\alpha}{2\pi^2}\psi'\left(\frac{\alpha}{\pi}+\frac{1}{2}\right).$$

i.e., $M_t = G\theta\, a^4\left\{\frac{\alpha}{4} - \frac{1}{\pi}\left[\psi\left(\frac{1}{2}+\frac{\alpha}{\pi}\right) - \psi\left(\frac{1}{2}\right)\right] + \frac{\alpha}{2\pi^2}\psi'\left(\frac{\alpha}{\pi}+\frac{1}{2}\right)\right\}$

When $\alpha = \pi$, due to the relations $\psi\left(\frac{3}{2}\right) - \psi\left(\frac{1}{2}\right) = 2$, $\psi'\left(\frac{3}{2}\right) = \frac{\pi^2}{2} - 4$

we have the known expression[3]

$$M_t = G\theta\, a^4\left(\frac{\pi}{2} - \frac{4}{\pi}\right).$$

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A TRIGONOMETRIC SERIES USED IN PHYSICAL PROBLEMS

(Joint work with Dr.B.S,Rama Krishna)

(Published in NATURE,Vol.171 (1953))

A TRIGONOMETRIC SERIES USED IN PHYSICAL PROBLEMS

In connexion with a boundary-value problem in room acoustics, we were led to consider the following trigonometric series. It has been noted further that certain other trigonometric series encountered by K.S.Krishnan[1,2] and L.S. Goddard[3] in different physical situations can be obtained as special cases of the above series.

The series encountered is:

$$\sum_{n=1}^{\infty} \frac{\sin(n+\theta)x}{n+\theta} \cdot \frac{\sin(n+\phi)x}{n+\phi} = A(\theta, \phi ; x), \text{ say, } \quad 0 < x < \pi,$$

and $A(\theta, \phi ; x)$ can be expressed as

$$\frac{1}{2}\frac{\cos(\theta-\phi)x}{\theta-\phi}\left[\psi(1+\theta) - \psi(1+\phi)\right] +$$

$$\frac{1}{2}\frac{\cos(\theta-\phi)x}{\theta-\phi}\left[c(\theta, 2x) - c(\phi, 2x)\right] +$$

$$\frac{1}{2}\frac{\sin(\theta-\phi)x}{\theta-\phi}\left[s(\theta, 2x) + s(\phi, 2x)\right],$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad c(\theta, x) = \sum_{n=1}^{\infty} \frac{\cos(n+\theta)x}{n+\theta},$$

$$s(\theta, x) = \sum_{n=1}^{\infty} \frac{\sin(n+\theta)x}{n+\theta}.$$

The transformations for $c(\theta, x)$ and $s(\theta, x)$ were obtained by G.H.Hardy[4].

Putting $\theta = \phi$ and taking the sum of $A(\theta, \theta ; x)$ and $A(-\theta, -\theta ; x)$, one obtains the result (Krishnan, loc. cit.):

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2(nx+\alpha)}{(nx+\alpha)^2} = \frac{\pi}{x}.$$

Putting $\theta = -\phi = \alpha m$ ($\alpha > 1$, $m = 1, 2, 3, \ldots$) and $x = \frac{\pi}{\alpha}$, one obtains (Goddard, loc. cit.):

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\pi/\alpha)}{n^2 - \alpha^2 m^2} = 0.$$

The corresponding series involving the cosines, namely,

$$\sum_{n=1}^{\infty} \frac{\cos(n+\theta)x}{n+\theta} \cdot \frac{\cos(n+\phi)x}{n+\phi} = B(\theta, \phi; x), \text{ say,}$$

can also be treated in the same manner.

The last series can be used to obtain the result

$$\sum_{n=-\infty}^{\infty} \frac{\cos^2(nx+\alpha)}{(nx+\alpha)^2} = \left(\frac{\pi}{x}\right)^2 \left[\operatorname{cosec}^2\left(\frac{\pi x}{\alpha}\right) - \frac{x}{\pi}\right].$$

The details of the calculations and certain other results will be published elsewhere.

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CERTAIN TRIGONOMETRIC SUMMATIONS

(Joint work with Dr. B.S.Rama Krishna)

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CERTAIN TRIGONOMETRIC SUMMATIONS

The trigonometric series considered below have been encountered in the course of certain investigations in room acoustics and in view of the occurrence of similar series in diverse physical problems, we consider it of some interest to present here an elementary method of expressing them in terms of standard functions. Using these transformations, it is shown that some results due to Krishnan[1] and Goddard[2] are derivable.

We consider the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n+\theta)x}{(n+\theta)} \cdot \frac{\sin(n+\phi)x}{(n+\phi)} , \qquad 0 < x < \pi \tag{1}$$

and the corresponding series in cosines

$$\sum_{n=1}^{\infty} \frac{\cos(n+\theta)x}{n+\theta} \cdot \frac{\cos(n+\phi)x}{n+\phi} , \qquad 0 < x < \pi \tag{2}$$

which are denoted here by $A(\theta, \phi; x)$ and $B(\theta, \phi; x)$ respectively.

We use the following known results[3]

(A) $$c(\theta, x) = \sum_{n=1}^{\infty} \frac{\cos(n+\theta)x}{n+\theta}$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\cos(\theta-\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt + \frac{1}{2}\left[\psi\left(\frac{1+\theta}{2}\right) - \psi\left(\frac{\theta}{2}\right)\right]\cos\pi\theta - \frac{\cos\theta x}{\theta}$$

(B) $$s(\theta, x) = \sum_{n=1}^{\infty} \frac{\sin(n+\theta)x}{n+\theta}$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\sin(\theta-\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt + \frac{1}{2}\left[\psi\left(\frac{1+\theta}{2}\right) - \psi\left(\frac{\theta}{2}\right)\right]\sin\pi\theta - \frac{\sin\theta x}{\theta}$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \lim_{n\to\infty}\left(\log n - \frac{1}{z} - \frac{1}{1+z} - \frac{1}{2+z} - \dots - \frac{1}{n+z}\right).$$

The series (1) can now be transformed thus

$$A(\theta, \phi; x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\cos(\theta-\phi)x - \cos[n+\frac{1}{2}(\theta+\phi)]2x}{(n+\theta)(n+\phi)}$$

$$= \frac{1}{2}\frac{\cos(\theta-\phi)x}{\theta-\phi}\sum_{n=1}^{\infty}\left(\frac{1}{n+\phi} - \frac{1}{n+\theta}\right)$$

$$+ \frac{1}{2}\,\frac{1}{\theta-\phi}\sum_{n=1}^{\infty}\left(\frac{1}{n+\theta} - \frac{1}{n+\phi}\right)\cos[n+\tfrac{1}{2}(\theta+\phi)]\,2x$$

Writing the terms $\frac{\cos[n+\frac{1}{2}(\theta+\phi)]2x}{n+\theta}$ and $\frac{\cos[n+\frac{1}{2}(\theta+\phi)]2x}{n+\phi}$

as $\frac{\cos[n+\theta-\frac{1}{2}(\theta-\phi)]2x}{n+\theta}$ and $\frac{\cos[n+\phi+\frac{1}{2}(\theta-\phi)]2x}{n+\phi}$ respectively,

$A(\theta,\phi;x)$ can be transformed as

$$A(\theta,\phi;x) = \frac{1}{2}\frac{\cos(\theta-\phi)x}{\theta-\phi}\left[\psi(1+\theta)-\psi(1+\phi)\right]$$
$$+\frac{1}{2}\frac{\cos(\theta-\phi)x}{\theta-\phi}\left[c(\theta,2x)-c(\phi,2x)\right]+\frac{1}{2}\frac{\sin(\theta-\phi)x}{\theta-\phi}\left[s(\theta,2x)+s(\phi,2x)\right]. \quad (3)$$

The series (2) can be written as

$$B(\theta,\phi;x) = \sum_{n=1}^{\infty}\frac{\cos(\theta-\phi)x - \sin(n+\theta)x\,\sin(n+\phi)x}{(n+\theta)(n+\phi)}$$
$$\cos(\theta-\phi)x\sum_{n=1}^{\infty}\frac{1}{(n+\theta)(n+\phi)} - A(\theta,\phi;x)$$

so that we have finally

$$B(\theta,\phi;x) = \frac{1}{2}\frac{\cos(\theta-\phi)x}{\theta-\phi}\left[\psi(1+\theta)-\psi(1+\phi)\right]$$
$$-\frac{1}{2}\frac{\cos(\theta-\phi)x}{\theta-\phi}\left[c(\theta,2x)-c(\phi,2x)\right]$$
$$-\frac{1}{2}\frac{\sin(\theta-\phi)x}{\theta-\phi}\left[s(\theta,2x)+s(\phi,2x)\right] \quad (4)$$

When $\theta=\phi$, $A(\theta,\theta;x)$ and $B(\theta,\theta;x)$ are easily obtained from (3) and (4) by passing to the limit. Then

$$A(\theta,\theta;x) = \frac{1}{2}\frac{d\psi(1+\theta)}{d\theta} + \frac{1}{2}\frac{d}{d\theta}c(\theta,2x) + x\,s(\theta,2x) \quad (3')$$

$$B(\theta,\theta;x) = \frac{1}{2}\frac{d\psi(1+\theta)}{d\theta} - \frac{1}{2}\frac{d}{d\theta}c(\theta,2x) - x\,s(\theta,2x) \quad (4')$$

We now derive certain trigonometric summations:

I.

$$\sum_{n=1}^{\infty}\left(\frac{\sin(n-\theta)x}{n-\theta}-\frac{\sin(n+\theta)x}{n+\theta}\right)\left(\frac{\sin(n-\phi)x}{n-\phi}-\frac{\sin(n+\phi)x}{n+\phi}\right)$$
$$=\pi\left(\frac{\sin(\theta-\phi)x}{\theta-\phi}-\frac{\sin(\theta+\phi)x}{\theta+\phi}\right) \quad (5)$$

II. Adding $A(\theta,\theta;x)$ and $A(-\theta,-\theta;x)$ we obtain

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 (nx+\alpha)}{(nx+\alpha)^2} = \frac{\pi}{x} \qquad (6)$$

where $\alpha = \theta x$, a result used by Krishnan in Light Scattering(loc.cit.)

III. Taking $\phi = -\theta$ in (5) we get

$$\sum_{n=1}^{\infty} \frac{\sin (n+\theta)x \sin (n-\theta)x}{n^2-\theta^2} = \frac{\sin^2 \theta x}{2\theta^2} + \frac{\pi}{4} \frac{\sin 2\theta x}{\theta}. \qquad (7)$$

In particular, when $\theta = \alpha m$ and $x = \frac{\pi}{\alpha}$, where α is real and > 1, and m is a positive integer, the above relation becomes

$$\sum_{n=1}^{\infty}{}' \frac{\sin^2 (n\pi/\alpha)}{n^2 - \alpha^2 m^2} = 0 \qquad (8)$$

a result used by Goddard(loc.cit.)

IV. By employing the transformation (4) we can obtain the following summation analogous to (5)

$$\sum_{n=1}^{\infty} \left(\frac{\cos (n-\theta)x}{n-\theta} - \frac{\cos (n+\theta)x}{n+\theta} \right) \cdot \left(\frac{\cos (n-\phi)x}{n-\phi} - \frac{\cos (n+\phi)x}{n+\phi} \right)$$

$$= \frac{1}{2} \left(\frac{1}{\theta} - \pi \cot \pi\theta \right) \left(\frac{\cos (\theta-\phi)x}{\theta-\phi} - \frac{\cos (\theta+\phi)x}{\theta+\phi} \right)$$

$$- \frac{1}{2} \left(\frac{1}{\phi} - \pi \cot \pi\phi \right) \left(\frac{\cos (\theta-\phi)x}{\theta-\phi} + \frac{\cos (\theta+\phi)x}{\theta+\phi} \right)$$

$$- \pi \left(\frac{\sin (\theta-\phi)x}{\theta-\phi} - \frac{\sin (\theta+\phi)x}{\theta+\phi} \right). \qquad (9)$$

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ON THE EVALUATION OF DIRICHLET'S INTEGRAL

(In the course of publication in the American Mathematical Monthly)

[redacted]

ON THE EVALUATION OF DIRICHLET'S INTEGRAL

The wellknown multiple integral

$$\iint_{(R_n)}\cdots\int x_1^{\alpha_1-1} x_2^{\alpha_2-1} \cdots x_n^{\alpha_n-1} (1-x_1-x_2-\cdots-x_n)^{\alpha_0-1}\, dx_1\, dx_2 \cdots dx_n$$

where R_n is the region defined by $x_1 \geqslant 0,\ x_2 \geqslant 0, \cdots, x_n \geqslant 0,\ \sum x_i \leqslant 1$ and $\alpha_0, \alpha_1, \cdots, \alpha_n$ are real positive constants is usually evaluated by the use of Dirichlet's transformation. The integral is expressed as a product of Beta functions and the procedure does not at any stage suggest the relation between the Beta and Gamma functions.

The following is an alternative procedure for evaluating the above integral by the use of Laplace Transform. It has the advantage that when $n = 1$ the procedure provides a proof of the relation between the Beta and Gamma functions.

Let $f_0(x), f_1(x), \cdots, f_n(x)$ be $n+1$ functions whose Laplace transforms viz., $\int_0^\infty e^{-px} f_n(x)\,dx\ (n = 0, 1, \cdots, n)$ exist. We have by the convolution theorem

(1) $L\,(f_0(x) * f_1(x) * f_2(x) * \cdots * f_n(x)) = L\,f_0(x) \cdot L\,f_1(x) \cdot L\,f_2(x) \cdots L\,f_n(x).$

Now $f_0(x) * f_1(x) * f_2(x) * \cdots * f_n(x)$ can be written as

$$\int_0^x f_n(x_n)\, dx_n \left[f_0(x) * f_1(x) * \cdots * f_{n-1}(x)\right]_{(x-x_n)}$$

$$= \cdots$$

$$= \int_0^x f_n(x_n)\, dx_n \int_0^{x-x_n} f_{n-1}(x_{n-1})\, dx_{n-1} \cdots \int_0^{x-x_n-x_{n-1}-\cdots-x_2} f_1(x_1)\, f_0(x-x_n-x_{n-1}-\cdots-x_2-x_1)\, dx_1$$

The inverse of (1) is

$$f_0(x) * f_1(x) * f_2(x) * \cdots * f_n(x) = L^{-1}\left\{L\,f_0(x) \cdot L\,f_1(x) \cdot L\,f_2(x) \cdots L\,f_n(x)\right\}.$$

Hence $\displaystyle\int_0^x f_n(x_n)\, dx_n \int_0^{x-x_n} f_{n-1}(x_{n-1})\, dx_{n-1} \cdots \int_0^{x-x_n-\cdots-x_2} f_1(x_1)\, f_0(x-x_1-\cdots-x_n)\, dx_1$

$$= L^{-1}\left\{L\,f_0(x) \cdot L\,f_1(x) \cdot L\,f_2(x) \cdots L\,f_n(x)\right\}.$$

(2) Now take $f_n(x) = x^{\alpha_n - 1}\ (n = 0, 1, \cdots\cdots, n),\ \alpha_n > 0$.

$$\int_0^x x_n^{\alpha_n-1}\, dx_n \int_0^{x-x_n} x_{n-1}^{\alpha_{n-1}-1}\, dx_{n-1} \cdots \int_0^{x-x_n-\cdots-x_2} x_1^{\alpha_1-1} (x-x_1-\cdots-x_n)^{\alpha_0-1}\, dx_1$$

$$= L^{-1}\left\{\frac{\Gamma\alpha_0}{p^{\alpha_0}} \cdot \frac{\Gamma\alpha_1}{p^{\alpha_1}} \cdots \frac{\Gamma\alpha_n}{p^{\alpha_n}}\right\} = \frac{\Gamma\alpha_0\, \Gamma\alpha_1 \cdots \Gamma\alpha_n}{\Gamma(\alpha_0+\alpha_1+\cdots+\alpha_n)}\, x^{\alpha_0+\alpha_1+\cdots+\alpha_n-1}$$

When $x = 1$ the above relation reduces to

$$\int_0^1 x_n^{\alpha_n - 1}\, dx_n \int_0^{1-x_n} x_{n-1}^{\alpha_{n-1}-1}\, dx_{n-1} \cdots \int_0^{1-x_n-\cdots-x_2} x_1^{\alpha_1 - 1}(1-x_1-\cdots-x_n)^{\alpha_0 - 1}\, dx_1 = \frac{\Gamma\alpha_0\,\Gamma\alpha_1 \cdots \Gamma\alpha_n}{\Gamma\alpha_0+\alpha_1+\cdots+\alpha_n}.$$

i.e.,

$$\iint_{(R_n)}\cdots\int x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}(1-x_1-\cdots-x_n)^{\alpha_0-1}\,dx_1\,dx_2\cdots dx_n = \frac{\Gamma\alpha_0\,\Gamma\alpha_1\cdots\Gamma\alpha_n}{\Gamma\alpha_0+\alpha_1+\cdots+\alpha_n}.$$

When $n = 1$ this can be written

$$\int_0^1 x_1^{\alpha_1-1}(1-x_1)^{\alpha_0-1}\,dx_1 = \frac{\Gamma\alpha_0\,\Gamma\alpha_1}{\Gamma\alpha_0+\alpha_1}\ ;\ \text{i.e., } B(\alpha_1, \alpha_0) = \frac{\Gamma\alpha_1\,\Gamma\alpha_0}{\Gamma\alpha_1+\alpha_0}.$$

If we have instead of (2) the choice

$$f_n(x) = x^{\alpha_n - 1}\quad (n = 1, 2, \cdots, n),\quad f_0(x) = f(x) = g(1-x)$$

we have in the same way the more general relation

$$\iint_{(R_n)}\cdots\int x_1^{\alpha_1-1}\,x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}\,g(x_1+x_2+\cdots+x_n)\,dx_1\cdots dx_n$$

$$= \frac{\Gamma\alpha_1\,\Gamma\alpha_2\cdots\Gamma\alpha_n}{\Gamma\alpha_1+\alpha_2+\cdots+\alpha_n}\int_0^1 t^{\alpha_1+\alpha_2+\cdots+\alpha_n-1}\,g(t)\,dt.$$

A PROPERTY OF COMPLEX MATRICES

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A PROPERTY OF COMPLEX MATRICES

Several authors[1,2] have noticed the following result concerning complex matrices.

A and B are real, square, symmetric matrices of the same order n. A is assumed to be positive semi definite, i.e., the quadratic form $x'Ax \geqslant 0$ for all real values of x. Then if $\det(A + iB) = 0$ there is a real vector satisfying the equation $(A + iB)z = 0$.

In this note we give some more information on the conditions and nature of the above result and also an extension of it.

The following is a proof of the above result due to Quade[2] and is included here for later use.

$\operatorname{Det}(A + iB) = 0$ so that there is a non-trivial complex vector $z = x + iy$ annihilated by $(A + iB)$. $(A+iB)(x+iy) = Ax - By + i(Ay + Bx) = 0$.

Hence $Ax = By, \quad Ay = -Bx.$ (1)

Then $x'Ax + y'Ay = x'By - y'Bx = 0$ (since B is symmetric)

Hence $x'Ax = y'Ay = 0$ (since A is positive semidefinite)

This requires that

$$Ax = Ay = 0 \tag{2}$$

and then from equation (1) we have

$$By = 0, \quad Bx = 0 \tag{3}$$

Hence $(A + iB)x = 0, \quad (A + iB)y = 0$

and either x or y is a non-trivial, real vector.

We can see that the above result is true also if A and B are symmetric and anyone of them is assumed positive (or, negative) semi-definite. From (2) and (3) clearly $\det A = 0, \det B = 0$.

Let now $(A + iB)$ be a complex matrix of the above desription with rank $n-1$

Then the vector solution $x+iy$ of $(A+iB)z=0$ is a real vector multiplied by a complex scalar. For, if not we must have x and y independent, so that the equation $ax+by=0$ implies $a=b=0$. Then $x+iy$, $x-iy$ are two independent complex vectors; i.e., the equation $(\alpha+i\beta)(x+iy)+(\gamma+i\delta)(x-iy)=0$ ($\alpha, \beta, \gamma, \delta$ real) implies $\alpha=\beta=\gamma=\delta=0$, and these two vectors are annihilated by $(A+iB)$, which cannot be the case as $(A+iB)$ has its rank $n-1$. Thus x and y are linearly dependent and both the matrices A and B are also of rank $n-1$.

Let us now take $(A+iB)$ to be of rank $n-r$. Then there are r linearly independent complex vectors $z_K = x_K + i y_K$ ($K=1,2,\ldots,r$) annihilated by $(A+iB)$. As in Quade's proof of the earlier result we have here again $Ax_K=0$, $Ay_K=0$, $Bx_K=0$, $By_K=0$. We can see immediately that of the $2r$ vectors x_K, y_K any $r+1$ are linearly dependent. If $\xi_1, \xi_2, \ldots, \xi_r, \xi_{r+1}$ are a linearly independent set of $r+1$ vectors we see that the $r+1$ complex vectors $\xi_1+i\xi_{r+1}, \xi_2+i\xi_{r+1}, \ldots \xi_r+i\xi_{r+1}, \xi_1-i\xi_{r+1}$ are also linearly independent and are all annihilated by the matrix $(A+iB)$, which clearly is not possible as $(A+iB)$ has rank $n-r$. If on the other hand a set of r vectors, linearly independent, is lacking (among the $2r$ vectors $x_1,\ldots,x_r, y_1,\ldots,y_r$) we cannot have the r complex vectors $x_1+iy_1, \ldots, x_r+iy_r$ linearly independent. Thus there must be a set of r <u>real</u> vectors, which are linearly independent and which are annihilated by the complex matrix $(A+iB)$. We may also see that both the matrices A and B have rank $n-r$.

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