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# CURVATURE PROPERTIES OF A RIEMANNIAN HYPERSURFACE

Thesis

Submitted for the Degree of

DOCTOR OF PHILOSOPHY

of the

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By

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# Preface

This Thesis is the result of an endeavour to add to the existing knowledge of the curvature properties of a Riemannian hypersurface, by generalising several well-known results and processes of the differential geometry of a surface in a Euclidean 3-space. A study of Riemannian Geometry clarified the inherent algebraic structure underlying the numerous and varied results of ordinary geometry and this thesis is intended as a humble contribution in that direction.

Paper I is preliminary, and is devoted to the construction of hypersurfaces and subspaces in Riemannian space, all of whose points are umbilics. Such spaces are frequently referred to in the subsequent papers.

In paper II, we start with a set of Riemannian spaces $V_n$, $V_{n+1}, \ldots, V_{n+p}$ each immersed in the succeeding one. Then considering a curve $C$ in $V_n$, the relations between the principal curvatures, the normal and geodesic curvatures of $C$ in the several spaces have been studied in some detail. The well known formula $\frac{1}{\rho} = \frac{\cos\omega}{R} + \frac{\cos\varpi}{\rho_g}$ is generalised so as to be applicable for any two of the above spaces. The generalisation is given by

$$\frac{\cos\omega_{q\sigma|}}{R_{q|}} + \frac{\cos\varpi_{q\sigma|}}{\rho_{q|}} = \frac{\cos\varpi_{q+1\sigma|}}{\rho_{q+1|}} \quad (\S 5, \text{paper II})$$

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The method of work in the first part of the paper has been, modified in the second part to study similar properties of a curve which lies in the variety of interesection of ~~two~~ mutually orthogonal hypersurfaces $V_n^1$, $V_n^2$ both of which lie in $V_{n+1}$

The intrinsic derivative of the unit normal to a hypersurface figures prominently in the study of its curvature ~~xxxxxx~~ properties, but as far as the author is aware, the successive intrinsic derivatives of this intrinsic derivative have not found any application so far. In paper III, a number of interesting applications of the second intrinsic derivative of the unit normal have been obtained. The expression $g^{ij} g^{pq} \Omega_{pi} \Omega_{qj}$ which is derived from Bianchi's third fundamental form by summing it in $n$ orthogonal directions is incidentally proved in this paper to be equal to the sum of the squares of the normal curvatures and the geodesic torsions of $n$ mutually orthogonal curves along $V_n$.

In papers IV, V and VI the analogues of some well-known images or representations met with in ordinary differential geometry have been introduced and a number of known results have been generalised and a few results may be new. In paper IV, the familiar spherical indicatrices of the tangents, principal normals and binormals of a skew curve have been generalised and called umbilical indicatrices. Paper V gives the umbilical representation of a hypersurface analogous to the spherical representation of a surface. An asymptotic line of $V_n$ is orthogonal to its image at the corresponding point, ~~while for a line of is orthogonal to its image at the corresponding point~~, while for a line of curvature, and its image, the tangents are codirectional. These are familiar results in ordinary space. Amongst some kof the other results may be mentioned the following. If $C$ and $\bar{C}$ are a line of curvature of $V_n$ and its umbilical image, the geodesic curvatures of $C$ in $V_n$ and the first, second etc., curvatures of $C$ in $V_{n+1}$ are proportional to the corresponures and the geodesic torsions of $n$ mutually orthogonal curves along $V_n$.

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The transformation of inversion in ordinary geometry is extended in paper VII, by means of the formula

$$\xi^\delta = \frac{c^2}{a_{\alpha\beta}y^\alpha y^\beta} y^\delta,$$

the expression $a_{\alpha\beta}y^\alpha y^\beta = r^2$ thus coming in the place of $r^2$
A hypersurface of umbilics "inverts" into a hypersurface of umbilics, two hypersurfaces intersect at the same angle as their inverses, and lines of curvature invert into lines of curvature.

Paper VIII is a small note wherein an expression for geodesic curvature is worked out interms of the Ricci coefficients of rotation and the angles made by the curve with an orthogonal ennuple of directions. The expression reduces for an ordinary surface to Liouville's formula.

$$k_g = \frac{d\theta}{ds} + \cos\theta\, k_{g|u} + \sin\theta\, k_{g|v}$$

The extensions of the differential operators $\nabla$, div and curl to a Riemannian $V_n$ are well known, and the extensions of many of the results of ordinary vector analysis are known. In paper IX is developed a "vector analysis" pertaining to vectors and tensors of a Riemannian $V_{n+1}$, the operators div. curl etc. however operating in a subspace $V_n$ . The idea is similar to the process of tensor differentiation (the semicolon process), whereby covariant differentiation is effected in a $V_n$ upon a hypersurface of umbilics inverts into a hypersurface of umbilics, two hypersurfaces intersect at the same angle as their inverses, and lines of curvature invert into lines of curvature.

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VolII to Riemannian hypersurfaces and subspaces. Just a few are given here, and the author hopes to devote some time to this work on a future occasion. A short bibliography is added at the end. The list includes mostly those papers and books which can be said to have some connection or other with curvature properties either in classical differential geometry or in Riemannian space. The bibliography is confined to such books and periodicals as I could get access to here.

Three of the papers comprising this Thesis have been submitted for publication. Paper I has been submitted to the Calcutta Mathematical Society. Paper III has been accepted for publication by the National Institute of Science, India. Paper IV will appear in the ~~October~~ November 1956 issue of the Proceedings of the Indian Academy of Sciences.

It is my very pleasant duty to record my grateful thanks to my Professor, Dr. C.N.Srinivasiengar Under whose guidance and scrutiny this Thesis has been worked out. By a special arrangement kindly made by the authorities of the University of Mysore, he continued to guide me even after retirement from official service. Without his painstaking guidance and scrutiny, this work would not have been accomplished. I cannot find adequate words to express my gratitude to him.

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facilities for research etc., and it is my earnest hope that I shall be considered to have deserved the assistance and encouragement rendered to me.

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