IX

Differential Invariants

Many of the familiar concepts and formulae of vector analysis in ordinary 3-space have been extended for a Riemannian space of n dimensions, but with suitable modifications. For example, the Cross-product of two vectors and the curl of a vector in ordinary space are vectors, but these are tensors of the second order in Riemannian Geometry. The definitions and formulae concerning vector analysis in Riemannian space will be found scattered in the text books and other references on Riemannian Geometry. These formulae are all confined to a specific V_n. An attempt is made in this paper to define gradient, divergence and curl in V_n for vectors or tensors defined in an enveloping space V_{n+1}. The idea is analogous to the process of what is known as tensor differentiation (the Semi-colon process) which gives a covariant derivative in V_n for tensors defined in V_{n+1}.

In section I, three operators $\nabla, \overline{\nabla}, \nabla^*$ are defined - operators in V_n acting on functions defined in V_{n+1} and their properties are studied. This may be compared with C.E.Weatherburn's definitions with the same nomenclature for a surface in ordinary space discussed in his Differential Geometry, Vol.I, Chapter XII, and Vol.II, §§ 19-20.

The Laguerre function and the Darboux function well known in the ordinary geometry of a surface in ordinary space are tensors of the second order in Riemannian Geometry. The definitions and formulae concerning vector analysis in Riemannian space will be found scattered in the text books and other references on Riemannian Geometry. These formulae are all confined to a specific V_n. An attempt is made in this paper to define gradient, divergence and curl in V_n for vectors or tensors defined in an enveloping space V_{n+1}. The idea is analogous to the process of what is known as tensor differentiation (the Semi-colon process) which gives a covariant derivative in V_n for tensors defined in V_{n+1}.

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The Laguerre function and the Darboux function well known in the ordinary geometry of a surface in ordinary space are extended to a Riemannian hypersurface, and a few properties have been noted. In section II, the curl in V_n of a vector or tensor u of V_{n+1}, and the divergence in V_n of a tensor of V_{n+1} are defined, and the consequential formulae noted. No need is felt to complicate the notations by writing $\nabla_{(n)}$, $\mathrm{curl}_{(n)}$, $\mathrm{div}_{(n)}$ etc. to denote the processes of this paper. We use the ordinary symbols ∇, curl, div for these operations, as the context will be sufficiently clear as to the functions on which these operate and the field in which they operate. Some important differences will be observed. Thus for a scalar function ϕ in V_n, the gradient $\nabla\phi$ defined in V_n is a covariant vector, but the gradient $\nabla\phi$ viz $\nabla_n(\phi)$ defined here gives a contravaniant vector of V_{n+1}. The curl of a gradient in V_n is known to be identically zero, but $\mathrm{curl}_{(n)}\nabla_n\phi$ defined here is not zero in general. These concepts have made it possible to extend to a Riemannian hypersurface some of the results of Weatherburn for an ordinary surface, and the subject appears to be capable of further extensive study.

Section I

2. The intrinsic derivative of a vector u^i w.r.t. a curve C in V_n is defined by the components $\frac{\delta u^i}{\delta s} = u^i_{,j} e^j$ where e^i are the components of the unit tangent vector to C.

The intrinsic derivative of a vector with components U^α in V_{n+1}, w.r.t. C is given by the components $\frac{\delta U^\alpha}{\delta s} = U^\alpha_{;j} e^j$.

Let us denote $T_{;j}$ by $\frac{\delta T}{\delta x^j}$ where T is a tensor of any order defined in V_{n+1}. $\frac{\delta}{\delta x^j}$ is an operatur which raises the degree of covariancy in V_n by one. In particular for a scalar function ϕ of V_n, $\frac{\delta\phi}{\delta x^j} = \frac{\partial\phi}{\partial x^j} = \phi_{;j}$

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Let us next define the operator

$$\nabla = y^\alpha_{;i} g^{ij} \frac{\delta}{\delta x^j} \qquad (1)$$

Thus if ϕ be a scalar point function of V_n,

$\nabla \phi = y^\alpha_{;i} g^{ij} \frac{\delta \phi}{\delta x^j}$ are the components of a contravariant vector, say V^α, in V_{n+1}.

We observe that $$V_\beta y^\beta_{;k} = a_{\beta\gamma} y^\gamma_{;i} g^{ij} \frac{\delta\phi}{\delta x^j} y^\beta_{;k}$$
$$= (g_{ik} g^{ij}) \frac{\delta\phi}{\delta x^j} = \phi_{,k} \quad \ldots \quad (2)$$

In general, for any tensor T of V_{n+1}

$\nabla T = y^\alpha_{;i} g^{ij} \frac{\delta T}{\delta x^j}$ is a tensor the order of contravaniancy of which is one degree higher than that of T. ∇ is thus an operator which raises the contravariancy of any tensor in V_{n+1} by one degree. Let e^i define a vector of V_n whose components in V_{n+1} are $E^\alpha = y^\alpha_{;i} e^i$.

Let us define $$\nabla . E = a_{\alpha\gamma} y^\alpha_{;i} g^{ij} \frac{\delta E^\gamma}{\delta x^j}$$
$$= a_{\alpha\gamma} y^\alpha_{;i} g^{ij} (y^\gamma_{;l} e^l)_{;j}$$
$$= (a_{\alpha\gamma} y^\alpha_{;i} y^\gamma_{;l}) g^{ij} e^l_{,j} + a_{\alpha\gamma} y^\alpha_{;i} g^{ij} y^\gamma_{;lj} e^l$$
$$= (g_{il} g^{ij}) e^l_{,j} + (a_{\alpha\gamma} y^\alpha_{;i} N^\gamma) g^{ij} \Omega_{jl} e^l$$
$= e^l_{,l}$ observing that $a_{\alpha\gamma} y^\alpha_{;i} N^\gamma = 0 \quad (i = 1, 2, \ldots, n)$

i.e. $\underline{\nabla . E = \operatorname{div}_n e}$ . . . . (3)

Therefore, the divergence of a vector e^i in V_n is equal to $\nabla . E$. This agrees with the well-known formula $\nabla . E = \operatorname{div} E$ in ordinary Geometry.

$$\nabla . (\phi E) = a_{\alpha\gamma} y^\alpha_{;i} g^{ij} \frac{\delta}{\delta x^j} (\phi E^\gamma)$$

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$$= a_{\alpha\gamma} y^\alpha_{;i} g^{ij} (y^\gamma_{;l} e^l)_{;j}$$

$$= (a_{\alpha\gamma} y^\alpha_{;i} y^\gamma_{;l}) g^{ij} e^l_{,j} + a_{\alpha\gamma} y^\alpha_{;i} g^{ij} y^\gamma_{;lj} e^l$$

$$= (g_{il} g^{ij}) e^l_{,j} + (a_{\alpha\gamma} y^\alpha_{;i} N^\gamma) g^{ij} \Omega_{jl} e^l$$

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$$\nabla . (\phi E) = a_{\alpha\gamma} y^\alpha_{;i} g^{ij} \frac{\delta}{\delta x^j} (\phi E^\gamma)$$

$$= a_{\alpha\gamma} y^\alpha_{;i} E^\gamma g^{ij} \frac{\delta \phi}{\delta x^j} + a_{\alpha\gamma} y^\alpha_{;i} g^{ij} E^\gamma_{;j} \phi .$$

i.e. $\underline{\nabla . (\phi E) = \nabla \phi . E + \phi (\nabla . E)}$ . . . . (4)

(4) is comparable with the well known formula

$$\operatorname{div} (\phi u) = \phi \operatorname{div} u + \nabla \phi . u \quad \text{in } V_n .$$

If f is a scalar point function of V_n,

$$\nabla f = y^\alpha_{;i} g^{ij} f_{;j} = P^\alpha \quad \text{(say)}.$$

$$\nabla^2 f = \nabla P^\alpha = y^\beta_{;l}\, g^{lm} P^\alpha_{;m} = y^\beta_{;l}\, g^{lm} \left(y^\alpha_{;i}\, g^{ij} f_{,j} \right)_{;m}$$

$$= y^\beta_{;l}\, g^{lm} \left[y^\alpha_{;im}\, g^{ij} f_{,j} + y^\alpha_{;i}\, g^{ij} f_{;jm} \right]$$

$$= y^\beta_{;l}\, g^{lm}\, \Omega_{im}\, g^{ij} f_{,j}\, N^\alpha + y^\beta_{;l}\, y^\alpha_{;i}\, g^{lm} g^{ij} f_{;jm}$$

$$\therefore\ a_{\alpha\beta} \nabla^2 f = g^{ij} f_{;ij} = \text{the Laplacian of } f \text{ in } V_n \qquad (5)$$

This may be compared with the result (12.11), p.63 in the book "Matrix and Tensor calculus' by Anistotle D.Michal.

The operator $\nabla\phi \cdot \nabla = a_{\alpha\beta} \left(y^\alpha_{;i}\, g^{ij} \frac{\delta\phi}{\delta x^j} \right) \left(y^\beta_{;l}\, g^{lm} \frac{\delta}{\delta x^m} \right)$

i.e. $$\nabla\phi \cdot \nabla = g^{lm} \frac{\delta\phi}{\delta x^l} \frac{\delta}{\delta x^m} \qquad (6)$$

This is an operator associated with the given scalar function ϕ and we shall denote this by the Symbol $\{\phi\}$.

$$\therefore\ \{\phi\}\psi = \nabla\phi \cdot \nabla\psi = g^{lm} \frac{\delta\phi}{\delta x^l} \frac{\delta\psi}{\delta x^m} = \{\psi\}\phi \qquad (7)$$

$\nabla\phi$, $\nabla\psi$ are contravariant vectors of V_{n+1}, but by (7) the value of the expression $\nabla\phi \cdot \nabla\psi$ is equal to the scalar product of the gradients of ϕ and ψ in V_n,
i.e. the vectors $\phi_{,l}$ and $\psi_{,m}$.

Next $\{\phi\} N = g^{lm} \frac{\delta\phi}{\delta x^l} \frac{\delta N}{\delta x^m} = g^{lm} \frac{\delta\phi}{\delta x^l} N^\nu_{;m}$

$$= -g^{lm} g^{pq}\, \Omega_{qm}\, y^\nu_{;p} \frac{\delta\phi}{\delta x^l}, \text{ since } N^\nu_{;m} = -g^{pq}\, \Omega_{qm}\, y^\nu_{;p} \quad [1]$$

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The operator $\nabla\phi\cdot\nabla = a_{\alpha\beta}\left(y^{\alpha}_{;i}\, g^{ij}\frac{\delta\phi}{\delta x^{j}}\right)\left(y^{\beta}_{;l}\, g^{ln}\frac{\delta}{\delta x^{m}}\right)$

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Next $\{\phi\}N = g^{lm}\frac{\delta\phi}{\delta x^{l}}\frac{\delta N}{\delta x^{m}} = g^{lm}\frac{\delta\phi}{\delta x^{l}}N^{\nu}_{;m}$

$= -g^{lm}g^{pq}\Omega_{qm}y^{\nu}_{;p}\frac{\delta\phi}{\delta x^{l}}$, since $N^{\nu}_{;m} = -g^{pq}\Omega_{qm}y^{\nu}_{;p}$ [1]

$$\therefore \underline{\{\phi\}N = -g^{lm}g^{pq}\Omega_{qm}y^{\nu}_{;p}\frac{\delta\phi}{\delta x^{l}} = -\overline{\nabla}\phi} \quad \text{(say)} \tag{8}$$

Thus from (8), we obtain a new operator

$$\underline{\overline{\nabla} = g^{lm}g^{pq}\Omega_{qm}y^{\nu}_{;p}\frac{\delta}{\delta x^{l}}} \tag{9}$$

Equation (8) can be compared with equation (1), chapter III, § 19 in "Differential Geometry" by C.E.Weatherburn, Vol.II.

Let us next define $\nabla^{*} = y^{\alpha}_{;l}\,\Omega^{lk}\frac{\delta}{\delta x^{k}} =$

~~contravariant~~ xxxxxx

[redacted]

a Contravariant operator, where Ω^{lk} is the associate tensor of Ω_{lk} in the usual sense.

$$\underline{\nabla^{*}\phi = y^{\alpha}_{;l}\,\Omega^{lk}\frac{\delta\phi}{\delta x^{k}}} \qquad (10)$$

= the components of a contravariant vector in $\mathcal{V}_{n+1}$.

$$\nabla^{*}\phi\cdot\nabla = g_{\alpha\gamma}\,y^{\alpha}_{;l}\,\Omega^{lk}\frac{\delta\phi}{\delta x^{k}}\,y^{\gamma}_{;i}\,g^{ij}\frac{\delta}{\delta x^{j}}$$

$$= \Omega^{lk}\left(g_{li}\,g^{ij}\right)\frac{\delta\phi}{\delta x^{k}}\frac{\delta}{\delta x^{j}}$$

$$\therefore\ \underline{\nabla^{*}\phi\cdot\nabla = \Omega^{lk}\frac{\delta\phi}{\delta x^{l}}\frac{\delta}{\delta x^{k}} = [\phi],\ (\text{say})} \qquad (11)$$

We observe that ∇, $\overline{\nabla}$, ∇^{*} are all operators defined for a $\mathcal{V}_{n}$ in $\mathcal{V}_{n+1}$, while $\{\phi\}$ and $[\phi]$ are operators associated with a given ϕ.

$$\left(\nabla^{*}\phi\cdot\nabla\right)N = \Omega^{lk}\frac{\delta\phi}{\delta x^{k}}\,N^{\gamma}_{;l}$$

$$= \Omega^{lk}\frac{\delta\phi}{\delta x^{k}}\left(-g^{mp}\,\Omega_{pl}\,y^{\gamma}_{;m}\right)$$

$$= -\left(\Omega^{lk}\,\Omega_{pl}\right)\frac{\delta\phi}{\delta x^{k}}\,g^{mp}\,y^{\gamma}_{;m}$$

$$= -g^{mk}\,y^{\gamma}_{;m}\frac{\delta\phi}{\delta x^{k}} = -\nabla\phi$$

Hence we have the result $\underline{\nabla^{*}\phi\cdot\nabla N = -\nabla\phi}$ (12)

In $\mathcal{V}_{n+1}$

$$\nabla^*\phi\cdot\nabla = g_{\alpha\gamma}\, y^{\alpha}_{,l}\, \Omega^{lk} \frac{\delta\phi}{\delta x^k}\, y^{\gamma}_{;i}\, g^{ij} \frac{\delta}{\delta x^j}$$

$$= \Omega^{lk}\,(g_{li}\,g^{ij})\,\frac{\delta\phi}{\delta x^k}\,\frac{\delta}{\delta x^j}$$

$$\therefore\ \nabla^*\phi\cdot\nabla = \Omega^{lk}\frac{\delta\phi}{\delta x^l}\frac{\delta}{\delta x^k} = [\phi], \text{ (say)} \qquad (11)$$

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Hence we have the result $\nabla^*\phi\cdot\nabla N = -\nabla\phi$ . . . (12)

Also, If ψ is another scalar point function,

$$\nabla^*\phi\cdot\nabla\psi = \Omega^{lk}\frac{\delta\phi}{\delta x^k}\frac{\delta\psi}{\delta x^l} = \nabla^*\psi\cdot\nabla\phi \qquad (13)$$

i.e. $[\phi]\psi = [\psi]\phi$

More generally, if T and Q are any two tensors it is easily verified ~~xxx~~ that

$$\nabla^*Q\cdot\nabla T = \Omega^{lk}\,Q_{;l}\,T_{;k} = \nabla^*T\cdot\nabla Q$$

$$= \nabla Q\cdot\nabla^*T = \nabla T\cdot\nabla^*Q \qquad (14)$$

i.e.

i.e. $[Q]T = [T]Q$

$$\overline{\nabla}\phi \cdot \nabla^{*}\psi = a_{\alpha\gamma}\, y^{\gamma}_{;p}\, g^{lm} g^{pq} \Omega_{qm} \frac{\delta\phi}{\delta x^{l}} y^{\alpha}_{;j} \Omega^{jk} \frac{\delta\psi}{\delta x^{k}}$$

$$= \left(a_{\alpha\gamma}\, y^{\alpha}_{;j}\, y^{\gamma}_{;p}\right) g^{pq} g^{lm} \Omega_{qm} \Omega^{jk} \frac{\delta\phi}{\delta x^{l}} \frac{\delta\psi}{\delta x^{k}}$$

$$= \left(g_{jp}\, g^{pq}\right) g^{lm} \Omega_{qm} \Omega^{jk} \frac{\delta\phi}{\delta x^{l}} \frac{\delta\psi}{\delta x^{k}}$$

$$= g^{lm} \frac{\delta\phi}{\delta x^{l}} \frac{\delta\psi}{\delta x^{m}}$$

$$\underline{\overline{\nabla}\phi \cdot \nabla^{*}\psi = \nabla^{*}\phi \cdot \overline{\nabla}\psi = \nabla\phi \cdot \nabla\psi} \qquad (15)$$

Similarly for any two tensors T **and** Q

$$\underline{\overline{\nabla}T \cdot \nabla^{*}Q = g^{lm} T_{;l}\, Q_{;m} = \overline{\nabla}Q \cdot \nabla^{*}T}$$

$$= \underline{\nabla^{*}T \cdot \overline{\nabla}Q = \nabla^{*}Q \cdot \overline{\nabla}T} \qquad (16)$$

$$\overline{\nabla}\phi \cdot \nabla\psi = a_{\alpha\gamma}\, g^{lm} g^{pq} \Omega_{qm}\, y^{\gamma}_{;p} \frac{\delta\phi}{\delta x^{l}} y^{\alpha}_{;j}\, g^{ij} \frac{\delta\psi}{\delta x^{i}}$$

$$= \left(a_{\alpha\gamma}\, y^{\alpha}_{;j}\, y^{\gamma}_{;p}\right)\left(g^{pq} g^{lm} \Omega_{qm} \frac{\delta\phi}{\delta x^{l}} \frac{\delta\psi}{\delta x^{i}} g^{ij}\right)$$

i.e. $$\underline{\overline{\nabla}\phi \cdot \nabla\psi = g^{lm} \Omega_{jm}\, g^{ij} \frac{\delta\phi}{\delta x^{l}} \frac{\delta\psi}{\delta x^{i}} = \overline{\nabla}\psi \cdot \nabla\phi} \qquad (17)$$

Similarly for any two tensors T **and** Q,

$$\underline{\overline{\nabla}T \cdot \nabla Q = g^{lm} \Omega_{jm}\, g^{ij}\, T_{;l}\, Q_{;i} = \overline{\nabla}Q \cdot \nabla T}$$

$$\underline{= \nabla T \cdot \overline{\nabla}Q = \nabla Q \cdot \overline{\nabla}T} \qquad (18)$$

$$\left(\sigma_{jp} g \cdots\right) \cdots \Omega_{qm} \cdots \frac{\delta\phi}{\delta x^l}\frac{\delta\psi}{\delta x^k}$$

$$= g^{lm} \frac{\delta\phi}{\delta x^l}\frac{\delta\psi}{\delta x^m}.$$

$$\underline{\overline{\nabla}\phi \cdot \nabla^*\psi = \nabla^*\phi \cdot \overline{\nabla}\psi = \nabla\phi \cdot \nabla\psi} \quad \ldots \quad (15)$$

Similarly for any two tensors T and Q

$$\underline{\overline{\nabla} T \cdot \nabla^* Q = g^{lm} T_{;l} Q_{;m} = \overline{\nabla} Q \cdot \nabla^* T}$$

$$= \underline{\nabla^* T \cdot \overline{\nabla} Q = \nabla^* Q \cdot \overline{\nabla} T} \quad \ldots \quad (16)$$

$$\overline{\nabla}\phi \cdot \nabla\psi = a_{\alpha\gamma}\, g^{lm} g^{pq} \Omega_{qm}\, y^{\gamma}_{;p} \frac{\delta\phi}{\delta x^l}\, y^{\alpha}_{;j}\, g^{ij} \frac{\delta\psi}{\delta x^i}$$

$$= \left(a_{\alpha\gamma}\, y^{\alpha}_{;j}\, y^{\gamma}_{;p}\right)\left(g^{pq} g^{lm} \Omega_{qm} \frac{\delta\phi}{\delta x^l}\frac{\delta\psi}{\delta x^i} g^{ij}\right)$$

i.e. $\underline{\overline{\nabla}\phi \cdot \nabla\psi = g^{lm} \Omega_{jm}\, g^{ij} \frac{\delta\phi}{\delta x^l}\frac{\delta\psi}{\delta x^i} = \overline{\nabla}\psi \cdot \nabla\phi}$ . . (17)

Similarly for any two tensors T and Q,

$$\underline{\overline{\nabla} T \cdot \nabla Q = g^{lm} \Omega_{jm}\, g^{ij} T_{;l}\, Q_{;i} = \overline{\nabla} Q \cdot \nabla T}$$

$$\underline{= \nabla T \cdot \overline{\nabla} Q = \nabla Q \cdot \overline{\nabla} T} \quad \ldots \quad (18)$$

<u>Taking any two of the operators $\nabla, \overline{\nabla}, \nabla^*$ they have thus a double commutative property w.r.t. the dot product operation on two tensors in that the positions of the operators can be interchanged, as also those of the tensors T and</u> Q.

3. We have C.E.Weatherburn's well-known result

$\nabla . N = \operatorname{div} N = -$ mean curvature of V_n

We shall similarly consider $\overline{\nabla} . N$ and $\nabla^* . N$.

$$\overline{\nabla}\cdot N = a_{\alpha\gamma}\, g^{lm} g^{pq}\Omega_{qm}\, y^{\gamma}_{;p}\, N^{\alpha}_{;l}$$

$$= \left(a_{\alpha\gamma}\, y^{\gamma}_{;p} N^{\alpha}_{;l}\right)\left(g^{lm} g^{pq}\Omega_{qm}\right)$$

$$\underline{\overline{\nabla}\cdot N = -g^{lm}\,\Omega_{lp}\,\Omega_{qm}\, g^{pq}} \qquad (19)$$

The expression on the right is ~~known as~~ derivable from Bianchi's third fundamental form $g^{ij}\Omega_{ip}\Omega_{jq}e^pe^q$ [2] for V and it has been proved in a previous paper [3] that the R.H.S. of (19) it is equal to the sum of the squares of the normal curvature and the squares of the geodesic torsion along the directions of any orthogonal ennuple of V_n.

In ordinary geometry, (19) reduces to Weatherburns formula

$$\overline{\nabla}\cdot N = -\left(\frac{L^2}{E^2} + \frac{N^2}{G^2}\right) = 2K - J^2.$$

Proof: Choosing the lines of curvature on an ordinary surface as parametric curves, we have $g_{pq} = 0 \therefore g^{pq} = 0 \; (p \neq q)$ and $\Omega_{pq} = 0 \; (p \neq q), \; (p, q = 1, 2)$

i.e. $F = 0$ and $M = 0$

$$\therefore g^{pq}\Omega_{pi}\,\Omega_{qj}\, g^{ij} = \sum_p \left(g^{pp}\Omega_{pp}\right)^2 = \left(\frac{L^2}{E^2} + \frac{N^2}{G^2}\right)$$

$$= \frac{2LN}{EG} + \frac{L^2}{E^2} + \frac{N^2}{G^2} - \frac{2LN}{EG}$$

$$= \left(\frac{L}{E} + \frac{N}{G}\right)^2 - 2\,\frac{LN}{EG}$$

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i.e. $F = 0$ and $M = 0$

$$\therefore g^{pq}\Omega_{pi}\Omega_{qj}g^{ij} = \sum_{p}(g^{pp}\Omega_{pp})^2 = \left(\frac{L^2}{E^2} + \frac{N^2}{G^2}\right)$$

$$= \frac{2LN}{EG} + \frac{L^2}{E^2} + \frac{N^2}{G^2} - \frac{2LN}{EG}$$

$$= \left(\frac{L}{E} + \frac{N}{G}\right)^2 - 2\frac{LN}{EG}$$

$$= J^2 - 2K.$$ where K is the Gaussian curvature and J is the mean curvature.

$$\therefore \underline{\overline{\nabla}\cdot N = 2K - J^2.}$$

Next, $$\nabla^{*}\cdot N = a_{\alpha\gamma}\, y^{\alpha}_{;l}\, \Omega^{lk} N^{\gamma}_{;k}$$

$$= (a_{\alpha\gamma}\, y^{\alpha}_{;l}\, \Omega^{lk})(-g^{mp}\Omega_{pk}\, y^{\gamma}_{;m})$$

$$= -(a_{\alpha\gamma}\, y^{\alpha}_{;l}\, y^{\gamma}_{;m})(g^{pm}\Omega^{lk}\Omega_{pk})$$

$$= -(g_{lm}g^{pm})(\Omega^{lk}\Omega_{pk})$$

$= -\Omega^{lk}\Omega_{lk} = -n$, where n is the dimension of the space.

$$\therefore \quad \underline{\nabla^* \cdot N = -n} \qquad (20)$$

In ordinary geometry, (20) reduces to the formula

$\nabla^* \cdot n = -2$ in Chapter III, $\S$20, p.32 of C.E.Weatherburn's Differential Geometry, Vol.II.

Similarly, if y denotes the vector given by the coordinates

$$y^\alpha, \ (\alpha = 1, 2, \ldots, n+1)$$

$$\underline{\nabla \cdot y = a_{\alpha\gamma}\, y^\alpha_{;i}\, g^{ij}\, y^\gamma_{;j} = g_{ij}\, g^{ij} = n} \qquad (21)$$

Compare with Ex.1, Chapter XII, $\S$ 120, p.233 of "Differential Geometry" by Weatherburn, Vol.I.

$$\overline{\nabla} \cdot y = a_{\alpha\gamma}\left(g^{lm} g^{pq}\Omega_{qm}\, y^\gamma_{;p}\right)\frac{\delta y^\alpha}{\delta x^l}$$

$$= \left(a_{\alpha\gamma}\, y^\alpha_{;l}\, y^\gamma_{;p}\right) g^{pq} g^{lm}\, \Omega_{qm} = g_{lp}\, g^{pq} g^{lm}\, \Omega_{qm}$$

i.e. $\underline{\overline{\nabla} \cdot y = g^{lm}\Omega_{lm} = -\nabla \cdot N =}$ the mean curvature M of V_n. (22)

This can be compared with Weatherburn's formula

$\overline{\nabla} \cdot r = J$, in Chapter III, $\S$ 20, p.31 of "Differential Geometry" by Weatherburn, Vol.II.

$$\underline{\nabla^* \cdot y = a_{\alpha\gamma}\, y^\alpha_{;l}\, \Omega^{lk}\, y^\gamma_{;k} = g_{lk}\, \Omega^{lk}} \qquad (23)$$

= the sum of the reciprocals of the normal curvature of V_n along any n mutually conjugate directions [4]

Similarly, if y denotes the vector given by the coordinates

$$y^\alpha, \ (\alpha = 1, 2, \ \ldots, n+1)$$

$$\nabla \cdot y = a_{\alpha\gamma}\, y^\alpha_{;i}\, g^{ij}\, y^\gamma_{;j} = g_{ij} g^{ij} = n \quad \ldots \ldots \quad (21)$$

Compare with Ex.1, Chapter XII, § 120, p.233 of "Differential Geometry" by Weatherburn, Vol.I.

$$\overline{\nabla} \cdot y = a_{\alpha\gamma}\left(g^{lm} g^{pq} \Omega_{qm}\, y^\gamma_{;p}\right) \frac{\delta y^\alpha}{\delta x^l}$$

$$= \left(a_{\alpha\gamma}\, y^\alpha_{;l}\, y^\gamma_{;p}\right) g^{pq} g^{lm} \Omega_{qm} = g_{lp}\, g^{pq} g^{lm} \Omega_{qm}$$

i.e. $\overline{\nabla} \cdot y = g^{lm} \Omega_{lm} = -\nabla \cdot N =$ the mean curvature M of V_n. (22)

This can be compared with Weatherburn's formula

$\overline{\nabla} \cdot r = J$, in Chapter III, § 20, p.31 of "Differential Geometry" by Weatherburn, Vol.II.

$$\nabla^* \cdot y = a_{\alpha\gamma}\, y^\alpha_{;l}\, \Omega^{lk}\, y^\gamma_{;k} = g_{lk} \Omega^{lk} \quad \ldots \ldots \quad (23)$$

= the sum of the reciprocals of the normal curvature of V_n along any n mutually conjugate directions [4]

This reduces in ordinary geometry, to Weathernurn's formula

$$\nabla^* \cdot r = \frac{J}{K}$$

Proof:- Choosing the lines of curvature as parametric curves, we have $g_{pq} = 0 \ (p \neq q)$ and $\Omega_{pq} = 0 \ (p \neq q) \ (p, q = 1, 2)$

i.e. $F = 0$ and $M = 0$

$$\therefore g_{lk} \Omega^{lk} = \sum_l g_{ll}/\Omega_{ll} = \frac{E}{L} + \frac{G}{N} = \frac{EN + GL}{LN}$$

$$= \frac{J}{K}$$ where J is the mean curvature and K = Gaussian curvature.

$$\nabla . \nabla y = a_{\alpha\beta} y^{\alpha}_{;l} g^{lk} \left(y^{\beta}_{;i} g^{ij} y^{\gamma}_{;j} \right)_{;k}$$

$$= g^{jk} \Omega_{jk} N^{\gamma} = M N^{\gamma} \quad (24)$$

$\therefore$ $\nabla . \nabla y = M N$ **where** M **is the mean curvature of** V_n .

$$\nabla^{*} . \nabla y = a_{\alpha\gamma} \Omega^{lk} y^{\alpha}_{;l} \left(y^{\gamma}_{;m} g^{m\beta} y^{\beta}_{;\beta} \right)_{;k}$$

$$= \left(a_{\alpha\gamma} y^{\alpha}_{;l} y^{\gamma}_{;m} \right) g^{m\beta} \Omega^{lk} y^{\beta}_{;\beta k} + a_{\alpha\gamma} \Omega^{lk} y^{\alpha}_{;l} y^{\gamma}_{;mk} g^{m\beta} y^{\beta}_{;\beta}$$

$$= g_{lm} g^{m\beta} \Omega^{lk} \Omega_{\beta k} N^{\beta} + \left(a_{\alpha\gamma} y^{\alpha}_{;l} N^{\gamma} \right) \left(\Omega_{m\beta} g^{m\beta} \Omega^{lk} y^{\beta}_{;\beta} \right)$$

$$= n N^{\beta} \quad \text{since} \quad a_{\alpha\gamma} y^{\alpha}_{;l} N^{\gamma} = 0$$

Thus $\nabla^{*} . \nabla y = n N$ (25)

But $\nabla . \nabla^{*} y = a_{\alpha\beta} y^{\alpha}_{;i} g^{ij} \left(y^{\beta}_{;l} \Omega^{lk} y^{\gamma}_{;k} \right)_{;j}$

$$= a_{\alpha\beta} y^{\alpha}_{;i} y^{\beta}_{;lj} \Omega^{lk} g^{ij} y^{\gamma}_{;k} + a_{\alpha\beta} y^{\alpha}_{;i} y^{\beta}_{;l} g^{ij} \Omega^{lk}_{;j} y^{\gamma}_{;k}$$

$$+ a_{\alpha\beta} y^{\alpha}_{;i} y^{\beta}_{;l} g^{ij} \Omega^{lk} y^{\gamma}_{;kj}$$

$$= \Omega^{jk}_{;j} y^{\gamma}_{;k} + \Omega^{jk} \Omega_{jk} N^{\gamma}$$

$$\therefore \nabla . \nabla^{*} y = \Omega^{jk}_{;j} y_{;k} + n N \quad (26)$$

We conclude that $\nabla . \nabla^{*} y \neq \nabla^{*} . \nabla y$. (27)

$$\nabla^* \nabla y = a_{\alpha\gamma}\, \Omega^{lk} y^\alpha_{;l} \left(y^\gamma_{;m}\, g^{m\beta} y^\beta_{;\beta} \right)_{;k}$$

$$= \left(a_{\alpha\gamma} y^\alpha_{;l}\, y^\gamma_{;m} \right) g^{m\beta}\, \Omega^{lk} y^\beta_{;\beta k} + a_{\alpha\gamma}\, \Omega^{lk} y^\alpha_{;l}\, y^\gamma_{;mk}\, g^{m\beta} y^\beta_{;\beta}$$

$$= g_{lm}\, g^{m\beta}\, \Omega^{lk}\, \Omega_{\beta k} N^\beta + \left(a_{\alpha\gamma} y^\alpha_{;l} N^\gamma \right) \left(\Omega_{m\beta}\, g^{m\beta}\, \Omega^{lk} y^\beta_{;\beta} \right)$$

$$= n N^\beta \quad \text{since} \quad a_{\alpha\gamma} y^\alpha_{;l} N^\gamma = 0$$

Thus $\underline{\nabla^* \cdot \nabla y = n N}$ . . . . (25)

But $\nabla \cdot \nabla^* y = a_{\alpha\beta}\, y^\alpha_{;i}\, g^{ij} \left(y^\beta_{;l}\, \Omega^{lk} y^\gamma_{;k} \right)_{;j}$

$$= a_{\alpha\beta}\, y^\alpha_{;i}\, y^\beta_{;lj}\, \Omega^{lk} g^{ij} y^\gamma_{;k} + a_{\alpha\beta}\, y^\alpha_{;i}\, y^\beta_{;l}\, g^{ij}\, \Omega^{lk}_{;j}\, y^\gamma_{;k}$$

$$+ a_{\alpha\beta}\, y^\alpha_{;i}\, y^\beta_{;l}\, g^{ij}\, \Omega^{lk} y^\gamma_{;kj}$$

$$= \Omega^{jk}_{,j}\, y^\gamma_{;k} + \Omega^{jk}\, \Omega_{jk} N^\gamma$$

$\therefore \underline{\nabla \cdot \nabla^* y = \Omega^{jk}_{,j}\, y_{;k} + n N}$ . . . . . . (26)

We conclude that $\underline{\nabla \cdot \nabla^* y \neq \nabla^* \cdot \nabla y}$ . . . . . . (27)

From equations (25) and (26), we have

$\underline{\left(\nabla \cdot \nabla^* y\right) \cdot N = \left(\nabla^* \cdot \nabla y\right) \cdot N = n}$ . . . . (28)

$$\bar{\nabla} \cdot \nabla y = a_{\alpha\gamma}\, g^{lm}\, g^{pq}\, \Omega_{qm}\, y^\gamma_{;p} \left[y^\alpha_{;i}\, g^{ij}\, y^\beta_{;j} \right]_{;l}$$

$$= \left(a_{\alpha\gamma} y^\alpha_{;i}\, y^\gamma_{;p} \right) g^{pq}\, g^{lm}\, \Omega_{qm}\, g^{ij}\, \Omega_{jl} N^\beta$$

$\underline{\bar{\nabla} \cdot \nabla y = g^{lm}\, \Omega_{mi}\, \Omega_{jl}\, g^{ij} N}$ . . . . . . (29)

$$\nabla\cdot\overline{\nabla}y = a_{\alpha\beta}\,y^{\beta}_{;i}\,g^{ij}\left(g^{lm}g^{pq}\Omega_{qm}\,y^{\alpha}_{;p}\,y^{\gamma}_{;l}\right)_{;j}$$

$$= \Omega_{im,j}\,g^{ij}g^{lm}\,y^{\alpha}_{;l} + g^{lm}\Omega_{mi}\Omega_{jl}\,g^{ij}N \quad \ldots \quad (30)$$

$$\therefore\ \nabla\cdot\overline{\nabla}y \neq \overline{\nabla}\cdot\nabla y \quad \ldots \quad (31)$$

But $(\overline{\nabla}\cdot\nabla y)\cdot N = (\nabla\cdot\overline{\nabla}y)\cdot N = g^{lm}\Omega_{lj}\Omega_{mi}\,g^{ij}$ . . . (32)

$$\overline{\nabla}\cdot\nabla^{*}y = a_{\alpha\beta}\,y^{\beta}_{;p}\,g^{pq}\Omega_{qm}\,g^{lm}\left(y^{\alpha}_{;i}\,\Omega^{ij}\,y^{\gamma}_{;j}\right)_{;l}$$

$$= g^{lj}\Omega_{lj}\,N^{\gamma} + g^{lm}\Omega_{mi}\,\Omega^{ij}_{,l}\,y^{\gamma}_{;j}$$

$$\overline{\nabla}\cdot\nabla^{*}y = MN + g^{lm}\Omega_{mi}\Omega^{ij}_{,l}\,y^{\gamma}_{;j} \quad \ldots \quad (33)$$

$$\nabla^{*}\cdot\overline{\nabla}y = MN + g^{lm}\Omega_{li,j}\,\Omega^{ij}\,y^{\gamma}_{;m} \quad \ldots \quad (34)$$

$$\therefore\ \nabla^{*}\cdot\overline{\nabla}y \neq \overline{\nabla}\cdot\nabla^{*}y \quad \ldots \quad (35)$$

But $(\overline{\nabla}\cdot\nabla^{*}y)\cdot N = (\nabla^{*}\cdot\overline{\nabla}y)\cdot N = M$ . . . (36)

Thus, unlike the result enunciated at the end of §2 , the dot product of any two of the operators $\nabla, \overline{\nabla}, \nabla^{*}$ is a non-commutative process, when applied to the vector y . But the commutative property is restored when the dot product of

$$\ldots \qquad (32)$$

$$\bar{\nabla}\cdot\nabla^{*}y = a_{\alpha\beta}\, y^{\beta}_{;p}\, g^{pq}\, \Omega_{qm}\, g^{lm}\left(y^{\alpha}_{;i}\,\Omega^{ij}\,y^{\gamma}_{;j}\right)_{;l}$$

$$= g^{lj}\,\Omega_{lj}\, N^{\gamma} + g^{lm}\,\Omega_{mi}\,\Omega^{ij}_{,l}\, y^{\gamma}_{;j}$$

$$\bar{\nabla}\cdot\nabla^{*}y = MN + g^{lm}\,\Omega_{mi}\,\Omega^{ij}_{,l}\, y^{\gamma}_{;j} \qquad (33)$$

$$\nabla^{*}\cdot\bar{\nabla}y = MN + g^{lm}\,\Omega_{li,j}\,\Omega^{ij}\, y^{\gamma}_{;m} \qquad (34)$$

$$\therefore \nabla^{*}\cdot\bar{\nabla}y \neq \bar{\nabla}\cdot\nabla^{*}y \qquad (35)$$

But $(\bar{\nabla}\cdot\nabla^{*}y)\cdot N = (\nabla^{*}\cdot\bar{\nabla}y)\cdot N = M \qquad (36)$

Thus, unlike the result enunciated at the end of $\S 2$, the dot product of any two of the operators $\nabla, \bar{\nabla}, \nabla^{*}$ is a non-commutative process, when applied to the vector y. But the commutative property is restored when the dot product of the whole is taken w.r.t.N.

For a hypersurface of umbilics, in virtue of the property $\Omega_{ij} = \lambda g_{ij}$ where $\lambda = M/n$ we have $\bar{\nabla} = \nabla^{*} = \frac{M}{n}\nabla$, and consequently, the inequalities (27), (31) and (35) become equalities.

4. $$\nabla\cdot\nabla N = a_{\alpha\gamma}\, y^{\alpha}_{;i}\, g^{ij}\left(y^{\gamma}_{;l}\, g^{lk}\, N^{\beta}_{;k}\right)_{;j}$$

$$= a_{\alpha\gamma}\, y^{\alpha}_{;i}\, g^{ij}\left[y^{\gamma}_{;lj}\, g^{lk}\, N^{\beta}_{;k} + y^{\gamma}_{;l}\, g^{lk}\, N^{\beta}_{;kj}\right]$$

$$= g_{il}\, g^{lk}\, N^{\beta}_{;kj}\, g^{ij}$$

$$= g^{ij}\, N^{\beta}_{;ij} = \left[-g^{lk}\,\Omega_{ki,j}\, y^{\beta}_{;l} - g^{lk}\,\Omega_{ki}\,\Omega_{jl}\, N^{\beta}\right] g^{ij}$$

$$\therefore\ \underline{N\cdot(\nabla\cdot\nabla N) = -g^{lk}\,\Omega_{ki}\,\Omega_{lj}\, g^{ij} = -\Omega^{l}_{i}\,\Omega^{i}_{l}} \qquad (37)$$

$$\underline{= \overline{\nabla}\cdot N \quad \text{by } (19)}$$

$$\nabla^{*}\cdot\nabla N = a_{\alpha\gamma}\,\Omega^{lk}\, y^{\gamma}_{;l}\left(y^{\alpha}_{;i}\, g^{ij}\, N^{\beta}_{;j}\right)_{;j}$$

$$= a_{\alpha\gamma}\,\Omega^{lk}\, y^{\gamma}_{;l}\left(y^{\alpha}_{;ik}\, g^{ij}\, N^{\beta}_{;j} + y^{\alpha}_{;i}\, g^{ij}\, N^{\beta}_{;jk}\right)$$

$$= \left(a_{\alpha\gamma}\, y^{\alpha}_{;i}\, y^{\gamma}_{;l}\right)\left(g^{ij}\,\Omega^{lk}\, N^{\beta}_{;jk}\right)$$

$$= \left(g_{il}\, g^{ij}\right)\Omega^{lk}\, N^{\beta}_{;jk} = \Omega^{jk}\, N^{\beta}_{;jk} = \Omega^{ij}\, N^{\beta}_{;ij}$$

$$= -\left[\Omega_{ki,j}\, g^{lk}\, y^{\beta}_{;l} + g^{lk}\,\Omega_{ki}\,\Omega_{jl}\, N^{\beta}\right]\Omega^{ij}$$

$$= -\left[g^{lj}\,\Omega_{lj}\, N^{\beta} + \Omega^{ij}\,\Omega_{ki,j}\, g^{lk}\, y^{\beta}_{;l}\right]$$

$$\therefore\ \underline{N\cdot(\nabla^{*}\nabla N) = -M = -\ \text{mean curvature of } \mathcal{V}_n} \qquad (38)$$

$$\therefore \quad \underline{N\cdot(\nabla\cdot\nabla N) = -g^{lk}\Omega_{ki}\Omega_{lj}g^{ij} = -\Omega^l_i\Omega^i_l} \qquad (37)$$

$$\underline{= \overline{\nabla}\cdot N \quad \text{by } (19)}$$

$$\nabla^*\cdot\nabla N = a_{\alpha\gamma}\Omega^{lk}y^\gamma_{;l}\left(y^\alpha_{;i}g^{ij}N^\beta_{;j}\right)_{;j}$$

$$= a_{\alpha\gamma}\Omega^{lk}y^\gamma_{;l}\left(y^\alpha_{;ik}g^{ij}N^\beta_{;j} + y^\alpha_{;i}g^{ij}N^\beta_{;jk}\right)$$

$$= \left(a_{\alpha\gamma}y^\alpha_{;i}y^\gamma_{;l}\right)\left(g^{ij}\Omega^{lk}N^\beta_{;jk}\right).$$

$$= \left(g_{il}g^{ij}\right)\Omega^{lk}N^\beta_{;jk} = \Omega^{jk}N^\beta_{;jk} = \Omega^{ij}N^\beta_{;ij}$$

$$= -\left[\Omega_{ki;j}g^{lk}y^\beta_{;l} + g^{lk}\Omega_{ki}\Omega_{jl}N^\beta\right]\Omega^{ij}$$

$$= -\left[g^{lj}\Omega_{lj}N^\beta + \Omega^{ij}\Omega_{ki;j}\,g^{lk}y^\beta_{;l}\right]$$

$$\therefore \quad \underline{N\cdot(\nabla^*\nabla N) = -M = -\text{ mean curvature of } V_n} \qquad (38)$$

$$\underline{= \nabla\cdot N \quad \text{by } (22)}$$

$$\overline{\nabla}\cdot\nabla N = a_{\alpha\beta}y^\beta_{;p}g^{pq}\Omega_{qm}g^{lm}\left(y^\alpha_{;i}g^{ij}N^\gamma_{;j}\right)_{;l}$$

$$= a_{\alpha\beta}y^\beta_{;p}g^{pq}\Omega_{qm}g^{lm}\left[y^\alpha_{;il}g^{ij}N^\gamma_{;j} + y^\alpha_{;i}g^{ij}N^\gamma_{;jl}\right]$$

$$= g^{jq}g^{lm}\Omega_{qm}N^\gamma_{;jl}$$

$$= g^{jq}g^{ml}\Omega_{qm}\left[-g^{\delta k}\Omega_{kj;l}y^\beta_{;\delta} - g^{\delta k}\Omega_{kj}\Omega_{l\delta}N^\beta\right]$$

$$\therefore\ N\cdot(\overline{\nabla}\cdot\nabla N) = -g^{\beta k}\Omega_{k\gamma}g^{\gamma q}\Omega_{qm}g^{ml}\Omega_{l\beta} \quad \ldots \quad (39)$$

$$= -\Omega^{\gamma}_{m}\Omega^{m}_{\beta}\Omega^{\beta}_{\gamma}$$

On a surface in ordinary space, this reduces to the sum of the cubes of the principal curvatures.

5. We shall define $E\cdot\nabla T = a_{\alpha\gamma}y^{\alpha}_{;i}e^{i}y^{\gamma}_{;l}g^{l\gamma}\dfrac{\delta T}{\delta x^{\gamma}}$ . . . (40)

where T is a tensor of any order;

and $E\cdot\dfrac{\delta G}{\delta s}\cdot\xi = a_{\alpha\beta}y^{\alpha}_{;\beta}e^{\beta}G^{\beta\gamma}_{;i}e^{i}a_{\gamma\delta}y^{\delta}_{;\gamma}\varepsilon^{\gamma}$ . . . . (41)

where ξ^{α} are the components of a vector in V_{n+1} whose components in V_n are ε^{i}, and G is a contravariant tensor of the second order in V_{n+1}.

Let $e_{1|}, e_{2|}, \ldots, e_{n|}$ be the unit vectors of an orthogonal ennuple of directions at a point of V_n.

$E_{h|}{}^{\alpha} = y^{\alpha}_{;i}\ e_{h|}{}^{i}$ are the components of $e_{h|}$ in the coordinates of V_{n+1}. $(h=1,2,\ldots,n)$

Let us evaluate,

$$D_{lhl} = E_{h|}\cdot\left(\frac{\delta}{\delta s_{h|}}\nabla N\right)\cdot E_{l|}$$

Now, $\nabla N = y^{\alpha}_{;i}\,g^{i\gamma}N^{\gamma}_{;\gamma}$ is a contravariant tensor of the second order, in V_{n+1}

$$\therefore\ \frac{\delta}{\delta s}(\nabla N) = \left(y^{\alpha}_{;i}\,g^{i\gamma}N^{\gamma}_{;i}\right)_{;l}e^{l}_{h|}$$

where T is a tensor of any order;

and $$E \cdot \frac{\delta G}{\delta s} \cdot \xi = a_{\alpha\beta} y^{\alpha}_{;i\xi} e^{\xi} G^{\beta\gamma}_{;i} e^{i} a_{\gamma\delta} y^{\delta}_{;j} \xi^{j} \quad . \quad . \quad . \quad (41)$$

where ξ^{α} are the components of a vector in V_{n+1} whose components in V_n are ξ^{i}, and G is a contravariant tensor of the second order in V_{n+1}.

Let $e_{1|}, e_{2|}, \quad . \quad . \quad . \quad , e_{n|}$ be the unit vectors of an orthogonal ennuple of directions at a point of V_n.

$E_{h|}^{\alpha} = y^{\alpha}_{;i} \; e_{h|}^{i}$ are the components of $e_{h|}$ in the coordinates of V_{n+1} $(h = 1, 2, \ldots, n)$

Let us evaluate,

$$D_{\lambda h l} = E_{h|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{l|}$$

Now, $\nabla N = y^{\alpha}_{;i} g^{ij} N^{\gamma}_{;j}$ is a contravariant tensor of the second order, in V_{n+1}

$$\therefore \frac{\delta}{\delta s_{h|}} (\nabla N) = \left(y^{\alpha}_{;i} g^{ij} N^{\gamma}_{;j}\right)_{;\xi} e^{\xi}_{h|}$$

$$= y^{\alpha}_{;i\xi} g^{ij} N^{\gamma}_{;j} e^{\xi}_{h|} + y^{\alpha}_{;i} g^{ij} N^{\gamma}_{;j\xi} e^{\xi}_{h|}$$

$$\therefore E_{h|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{l|} = a_{\alpha\beta} y^{\beta}_{;p} e^{p}_{l|} \left[\Omega_{i\xi} g^{ij} N^{\alpha} e^{\xi}_{h|} N^{\gamma}_{;j}\right.$$

$$\left. + y^{\alpha}_{;i} g^{ij} N^{\gamma}_{;j\xi} e^{\xi}_{h|}\right] a_{\delta\gamma} y^{\delta}_{;q} e^{q}_{l|}$$

$$= g_{pi} g^{ij} N^{\gamma}_{;j\xi} e^{\xi}_{h|} a_{\delta\gamma} y^{\delta}_{;q} e^{q}_{l|} e^{p}_{h|}$$

$$= \left(a_{\delta\nu} N^{\nu}_{;p\ell}\, y^{\delta}_{;q}\right) e_{h|}{}^{\ell} e_{l|}{}^{q} e_{h|}{}^{p}$$

But $\quad a_{\delta\nu} N^{\nu}_{;p}\, y^{\delta}_{;q} = -\Omega_{pq}$

$$\therefore\ a_{\delta\nu} N^{\nu}_{;p\ell}\, y^{\delta}_{;q} + a_{\delta\nu} N^{\nu}_{;p}\, y^{\delta}_{;\ell q} = -\Omega_{pq,\ell}$$

$\therefore\ a_{\delta\nu} N^{\nu}_{;p\ell}\, y^{\delta}_{;q} = -\Omega_{pq,\ell}$ since $a_{\delta\nu} N^{\nu}_{;p}\, y^{\delta}_{;q\ell} = \Omega_{q\ell}\, a_{\delta\nu} N^{\delta} N^{\nu}_{;p} =$

Hence finally $\quad E_{h|} \cdot \left(\dfrac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{l|} = -\Omega_{pq,\ell}\, e_{l|}{}^{p} e_{h|}{}^{\ell} e_{l|}{}^{q}$

Similarly $\quad E_{l|} \cdot \left(\dfrac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{h|} = -\Omega_{pq,\ell}\, e_{l|}{}^{p} e_{h|}{}^{\ell} e_{h|}{}^{q}$

$$= -\Omega_{qp,\ell}\, e_{h|}{}^{q} e_{l|}{}^{p} e_{h|}{}^{\ell} = E_{h|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{l|}$$

Hence we have the result

$$\underline{E_{h|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{l|} = E_{l|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{h|}} \qquad (42)$$

i.e. $\quad \underline{D_{hhl} = D_{lhh}}$

In ordinary geometry, (42) gives the results

$$\left.\begin{aligned} a \cdot \left(\frac{d}{ds} \nabla n\right) \cdot b &= b \cdot \left(\frac{d}{ds} \nabla n\right) \cdot a \\ \text{and}\quad b \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot a &= a \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot b \end{aligned}\right\} [5] \qquad (43)$$

where a and b are unit orthogonal surface vectors and ds, ds'

$\therefore\ a_{\delta\gamma} N^{\gamma}_{;p\beta} y^{\delta}_{;q} = -\Omega_{pq,\beta}$ since $a_{\delta\gamma} N^{\gamma}_{;p} y^{\delta}_{;q\beta} = \Omega_{q\beta}\, a_{\delta\gamma} N^{\delta} N^{\gamma}_{;p}$

Hence finally $E_{h|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{l|} = -\Omega_{pq,\beta}\, e_{h|}^{p} e_{h|}^{\beta} e_{l|}^{q}$

Similarly $E_{l|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{h|} = -\Omega_{pq,\beta}\, e_{l|}^{p} e_{h|}^{\beta} e_{h|}^{q}$

$$= -\Omega_{qp,\beta}\, e_{h|}^{q} e_{l|}^{p} e_{h|}^{\beta} = E_{h|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{l|}$$

Hence we have the result

$$\underline{E_{h|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{l|} = E_{l|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{h|}} \quad \cdot \quad \cdot \quad \cdot \quad (42)$$

i.e. $\underline{D_{hhl} = D_{lhh}}$

In ordinary geometry, (42) gives the results

$$\left.\begin{aligned} a \cdot \left(\frac{d}{ds} \nabla n\right) \cdot b &= b \cdot \left(\frac{d}{ds} \nabla n\right) \cdot a \\ \text{and}\quad b \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot a &= a \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot b \end{aligned}\right\} [5] \qquad (43)$$

where a and b are unit orthogonal surface vectors and ds, ds' are respectively the arc elements in the directions a and b .

We have $D_{hlh} = E_{h|} \cdot \left(\frac{\delta}{\delta s_{l|}} \nabla N\right) \cdot E_{h|} = -\Omega_{pq,\beta}\, e_{h|}^{p} e_{l|}^{q} e_{h|}^{q}$

$$= -\Omega_{\beta q,p}\, e_{l|}^{p} e_{h|}^{q} e_{h|}^{\beta} = -\Omega_{q\beta,p}\, e_{l|}^{p} e_{h|}^{q} e_{h|}^{\beta}$$

$$\therefore\ E_{h|} \cdot \left(\frac{\delta}{\delta s_{l|}} \nabla N\right) \cdot E_{h|} - E_{l|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{h|}$$

$$= \left(\Omega_{qp,\beta} - \Omega_{q\beta,p}\right) e_{l|}^{p} e_{h|}^{q} e_{h|}^{\beta}$$

$$\text{L.H.S.} = \underline{\overline{R}_{\alpha\beta\gamma\delta}\, y^{\alpha}_{;q}\, y^{\beta}_{;p}\, y^{\gamma}_{;\beta}\, N^{\delta} e_{l|}^{p} e_{h|}^{q} e_{h|}^{\beta}} \quad \cdot \quad \cdot \quad \cdot \quad (44)$$

by the Mainardi Codazzi identities, where $\overline{R}_{\alpha\beta\gamma\delta}$ is the Riemann-Christoffel tensor of V_{n+1}. [6]

If V_{n+1} is a Euclidean space, the m right hand side of (44) is zero.

For a hypersurface in a Euclidean space

$$\underline{E_{h|} \cdot \left(\frac{\delta}{\delta s_{l|}} \nabla N\right) \cdot E_{h|} = E_{l|} \cdot \left(\frac{\delta}{\delta s_{h|}} \dot{\nabla} N\right) \cdot E_{h|}} \quad \ldots (45)$$

i.e. $\underline{D_{hlh} = D_{lhh}}$

In ordinary geometry, (45) becomes

$$\left.\begin{aligned} a \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot a &= b \cdot \left(\frac{d}{ds} \nabla n\right) \cdot a \\ \text{and} \quad b \cdot \left(\frac{d}{ds} \nabla n\right) \cdot b &= a \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot b \end{aligned}\right\} [7] \quad \ldots (46)$$

Combining (43) and (46),

$$\left.\begin{aligned} a \cdot \left(\frac{d}{ds} \nabla n\right) \cdot b = b \cdot \left(\frac{d}{ds} \nabla n\right) \cdot a = a \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot a = -D \\ b \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot a = a \cdot \left(\frac{d}{ds'} \nabla n\right) \cdot b = b \cdot \left(\frac{d}{ds} \nabla n\right) \cdot b = D' \end{aligned}\right\} [8$$

where D and D' are the Darboux functions for the directions a and b respectively.

In analogy with the definitions of the Darboux function and the Laguerre function for a surface in ordinary space, we define $D_{hlh} = E_{h|} \cdot \left(\frac{\delta}{\delta s_{l|}} \nabla N\right) \cdot E_{h|}$ as the Darboux function of the direction $e_{h|}$ w.r.t. the direction $e_{l|}$, $h \neq l$. When $h = l$

$$E_{h|}\cdot\left(\frac{\delta}{\delta s_{l|}}\nabla N\right)\cdot E_{h|} = E_{l|}\cdot\left(\frac{\delta}{\delta s_{h|}}\nabla N\right)\cdot E_{h|} \quad \text{. . . (45)}$$

i.e. $D_{hlh} = D_{lhh}$

In ordinary geometry, (45) becomes

$$\left.\begin{aligned} a\cdot\left(\frac{d}{ds'}\nabla n\right)\cdot a &= b\cdot\left(\frac{d}{ds}\nabla n\right)\cdot a \\ \text{and}\quad b\cdot\left(\frac{d}{ds}\nabla n\right)\cdot b &= a\cdot\left(\frac{d}{ds'}\nabla n\right)\cdot b \end{aligned}\right\} [7] \quad \text{. . . (46)}$$

Combining (43) and (46),

$$\left.\begin{aligned} a\cdot\left(\frac{d}{ds}\nabla n\right)\cdot b &= b\cdot\left(\frac{d}{ds}\nabla n\right)\cdot a = a\cdot\left(\frac{d}{ds'}\nabla n\right)\cdot a = -D \\ b\cdot\left(\frac{d}{ds'}\nabla n\right)\cdot a &= a\cdot\left(\frac{d}{ds'}\nabla n\right)\cdot b = b\cdot\left(\frac{d}{ds}\nabla n\right)\cdot b = D' \end{aligned}\right\} [8]$$

where D and D' are the Darboux functions for the directions a and b respectively.

In analogy with the definitions of the Darboux function and the Laguerre function for a surface in ordinary space, we define $D_{hlh} = E_{h|}\cdot\left(\frac{\delta}{\delta s_{l|}}\nabla N\right)\cdot E_{h|}$ as the Darboux function of the direction $e_{h|}$ w.r.t. the direction $e_{l|}$, $h \neq l$. When $h=l$, this is the Laguerre function for the direction $e_{h|}$.

We difine a Darboux line as a curve in V_n for which the tangent vector $e_{h|}$ satisfied the $(n-1)$ equations $D_{hlh}=0$ ($l=1,2,\ldots,n$, $l\neq h$). Similarly a Laguerre line is defined as a curve in V_n for which the tangent vector $e_{h|}$ satisfied the equation $D_{hhh}=0$.

The sum of the Darboux functions of a direction ξ w.r.t. n mutually orthogonal directions of an ennuple $e_{h|}$ $(h=1,2,\ldots,n)$ at a point of V_n

$$= \sum_h \xi_{h|} \cdot \left(\frac{\delta}{\delta\sigma} \nabla N\right) \cdot \xi_{h|} = -\sum_h \Omega_{q\ell,p}\, e_{h|}^{q} e_{h|}^{\ell}\, \xi^{p}$$

$$= -\left(g^{q\ell}\,\Omega_{q\ell}\right)_{;p} \xi^{p} = -M_{,p}\,\xi^{p} \quad \ldots \quad (47)$$

where $d\sigma$ is the differential of the arc in the direction of ξ.

Hence, we have the theorem:

The sum of the Darboux functions of a direction ξ, w.r.t. any n mutually orthogonal directions of V_n is numerically equal to the intrinsic derivative of the mean curvature in the direction ξ.

In particular, choosing the vector ξ as one of the directions of the ennuple itself, the above theorem reduces to the following:

The sum of the Laguerre function for the direction $e_{1|}$ and the Darboux functions of $e_{1|}$ w.r.t. the remaining $(n-1)$ orthogonal directions of the ennuple is equal to the intrinsic derivative of the mean curvature in the direction $e_{1|}$.

This result reduces in ordinary space to the following known theorem:

where ds is the differential of the arc in the direction of ξ .

Hence, we have the theorem:

The sum of the Darboux functions of a direction ξ, w.r.t. any n mutually orthogonal directions of V_n is numerically equal to the intrinsic derivative of the mean curvature in the direction ξ .

In particular, choosing the vector ξ as one of the directions of the ennuple itself, the above theorem reduces to the following:

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This result reduces in ordinary space to the following known theorem:

The sum of the Darboux function of any direction and the Laguerre function of the perpendicular direction is a linear function, equal to twice the rate of change in the latter direction of the mean curvature of the surface. [10]

$$D_{hhh} = E_{h|} \cdot \left(\frac{\delta}{\delta s_{h|}} \nabla N\right) \cdot E_{h|} = -\Omega_{pq,l}\, e_{h|}^{p} e_{h|}^{q} e_{h|}^{l}$$

$$= \frac{\delta}{\delta s_{h|}}\left(\Omega_{pq}\, e_{h|}^{p} e_{h|}^{q}\right) - 2\,\Omega_{pq}\, e_{h|}^{p} p_{h|}^{q} \frac{1}{\rho_{g|h}} \quad \cdots (48)$$

where $e^{a}_{h|;b}\, e^{b}_{h|} = \dfrac{p_{h|}{}^{a}}{b_{g|h}}$ = principal normal vector.

From (48) we have the result: A geodesic of constant normal curvature or a line of curvature of constant normal curvature is always a Laguerre line. For $\frac{1}{b_{g|h}} = 0$ for a geodesic and $\Omega_{ij}\, e_{h|}{}^{i} p_{h|}{}^{j} = 0$ for a line of curvature. [9]

In ordinary geometry, (48) reduces to $L = \frac{d}{ds} k_n - 2\tau\gamma$, where $L =$ Laguerre function, $k_n =$ normal curvature, $\gamma =$ geodesic curvature and $\tau =$ geodesic torsion.

<u>Proof</u>: Let $e^{i}, p^{i}\ (i = 1, 2)$ be respectively the unit tangent vector and the unit principal normal vector to a curve C in a surface in the Euclidean space S_3.

$y^{\alpha}_{;i} e^{i},\ y^{\alpha}_{;i} p^{i},\ N^{\alpha}\ (\alpha = 1, 2, 3)$ are mutually orthogonal unit vectors of S_3.

Therefore, in virtue of the Serret-Frenet formula we have,

$$\frac{\delta N^{\alpha}}{\delta s} = \tau\, y^{\alpha}_{;i} p^{i} - k_n\, y^{\alpha}_{;i} e^{i} \qquad (49)$$

From (49) we get

$$\tau = \frac{\delta N^{\alpha}}{\delta s} \cdot y^{\alpha}_{;i} p^{i}$$

But, since $N^{\alpha} \cdot y^{\alpha}_{;i} p^{i} = 0$ we have $\frac{\delta N^{\alpha}}{\delta s} \cdot y^{\alpha}_{;i} p^{i}$

$$= -N^{\alpha} \cdot \frac{\delta}{\delta s}\left(y^{\alpha}_{;i} p^{i}\right)$$

Proof: Let $e^i, p^i\ (i=1,2)$ be respectively the unit tangent vector and the unit principal normal vector to a curve C in a surface in the Euclidean space S_3.

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$$= -N^\alpha\cdot\frac{\delta}{\delta s}\left(y^\alpha_{;i}p^i\right)$$

$$\therefore \tau = -N^\alpha\cdot\frac{\delta}{\delta s}\left(y^\alpha_{;i}p^i\right)$$

$$= -N^\alpha\left[y^\alpha_{;ij}\,e^i p^j + \frac{\delta p^i}{\delta s}\,y^\alpha_{;i}\right]$$

$$\tau = -\Omega_{ij}\,e^i p^j \qquad (50)$$

From (48) $L = \frac{d}{ds}k_n - 2\tau\sqrt{\ }$ This is the same as the expression (28), Chapter VIII, §86, p.139 in "Differential Geometry" by Weatherburn, Vol.II.

On a hypersurface of umbilics, the Laguerre function in any direction is equal to $\frac{1}{n}$ of the intrinsic derivative of the mean curvature at the point in the direction considered.

On a totally geodesic hypersurface, every line is a Darboux line as well as a Laguerre line.

SECTION II

1. Let us define the curl of a vector with components U^{α} in V_{n+1} at points of V_n as given by the components

$$\left(U^{\alpha}_{;k} g^{lk} y^{\beta}_{;l} - U^{\beta}_{;k} g^{lk} y^{\alpha}_{;l}\right)$$

This is a contravariant tensor of V_{n+1} of the second order and will be denoted by $\nabla \times U$. The definition is readily extended to any tensor T of order $p \geq 1$. The tensor $U^{\alpha}V^{\beta} - U^{\beta}V^{\alpha}$ formed by the vectors U and V is denoted by the cross product $U \times V = T^{\alpha\beta}$ (say). It is obvious that $U \times V$ is a skew symmetric tensor, and that $a_{\alpha\beta} T^{\alpha\beta} = 0$. In fact if $Q_{\alpha\beta}$ is any symmetric covariant tensor of the second order, $Q_{\alpha\beta} T^{\alpha\beta} = 0$.

The curl defined above should be distinguished from curlu in which is the covariant fensor with components $(u_{i,j} - u_{j,i})^{*}$ [11]

SECTION II

1. Let us define the curl of a vector with components U^{α} in $V_{\lambda+1}$ at points of V_{λ} as given by the components

$$\underline{\left(U^{\alpha}_{;k}\, g^{lk}\, y^{\beta}_{;l} - U^{\beta}_{;k}\, g^{lk}\, y^{\alpha}_{;l} \right)}$$

This is a contravariant tensor of V_{n+1} of the second order and will be denoted by $\nabla \times U$. The definition is readily extended to any tensor T of order $p \geq 1$. The tensor $U^{\alpha}V^{\beta} - U^{\beta}V^{\alpha}$ formed by the vectors U and V is denoted by the cross product $U \times V = T^{\alpha\beta}$ (say). It is obvious that $U \times V$ is a skew symmetric tensor, and that $a_{\alpha\beta}T^{\alpha\beta} = 0$. In fact if $Q_{\alpha\beta}$ is any symmetric covariant tensor of the second order, $Q_{\alpha\beta}T^{\alpha\beta} = 0$. The curl defined above should be distinguished from curlu in which is the covariant tensor with components $(u_{i,j} - u_{j,i})^{*}$ [11]

We shall define the swx square of the magnitude of a tensor T of the second order as

$$= \frac{1}{2} a_{\alpha\gamma} a_{\beta\delta} T^{\alpha\beta} T^{\gamma\delta};$$

and the scalar product of two tensors $P^{\alpha\beta}, Q^{\gamma\delta}$ as $\frac{1}{2} a_{\alpha\gamma} a_{\beta\delta} P^{\alpha\beta} Q^{\gamma\delta}$.

---

* This definition differs in sign from the usual definition of curl in ordinary vector analysis. Hence we find slight differences in the consequential formulae between the results here and in ordinary space.

According to the above definition if U and V are unit vectors,

$$(\text{Magnitude})^2 \text{ of } U\times V = \frac{1}{2} a_{\alpha\gamma} a_{\beta\delta} (U^\alpha V^\beta - U^\beta V^\alpha)(U^\gamma V^\delta - U^\delta V^\gamma)$$

$$= 2\left(\frac{1}{2} a_{\alpha\gamma} U^\alpha U^\gamma a_{\beta\delta} V^\beta V^\delta - \frac{1}{2} a_{\alpha\gamma} V^\alpha U^\gamma a_{\beta\delta} U^\beta V^\delta\right)$$

i.e. $(\text{Mag})^2$ of $(U\times V) = 1 - \cos^2\theta = \sin^2\theta$, where θ is the angle between the vectors U and V.

<u>It follows as in ordinary geometry that the magnitude of the cross product of two vectors is equal to the product of the magnitudes of the vectors multiplied by $\sin\theta$</u> . . . . . . (1)

Next we have,

$$\text{Curl}(\phi U) = \left\{(\phi U^\alpha)_{;i} g^{i\gamma} y^\beta_{;\gamma} - (\phi U^\beta)_{;i} g^{i\gamma} y^\alpha_{;\gamma}\right\}$$

$$= \phi_{;i}\left[U^\alpha g^{i\gamma} y^\beta_{;\gamma} - U^\beta g^{i\gamma} y^\alpha_{;\gamma}\right] + \phi\left[U^\alpha_{;i} g^{i\gamma} y^\beta_{;\gamma} - U^\beta_{;i} g^{i\gamma} y^\alpha_{;\gamma}\right]$$

$$\therefore \underline{\text{Curl}(\phi U) = U\times\nabla\phi + \phi\,\text{Curl}\,U} \quad \ldots\ldots \quad (2)$$

This agrees with the corresponding formula in E_n.

$$\text{Curl}\,\nabla\phi = (y^\alpha_{;\gamma} g^{\gamma i}\phi_{;i})_{;k} g^{kl} y^\beta_{;l} - (y^\beta_{;\gamma} g^{\gamma i}\phi_{;i})_{;k} g^{kl} y^\alpha_{;l}$$

$$= (y^\alpha_{;\gamma k} g^{\gamma i}\phi_{;i} + y^\alpha_{;\gamma} g^{\gamma i}\phi_{;ik}) g^{lk} y^\beta_{;l}$$

$$- (y^\beta_{;\gamma k} g^{\gamma i}\phi_{;i} + y^\beta_{;\gamma} g^{\gamma i}\phi_{;ik}) g^{kl} y^\alpha_{;l}$$

It follows as in ordinary geometry that the magnitude of the cross product of two vectors is equal to the product of the magnitudes of the vectors multiplied by $\sin\theta$ . . . . . . (1)

Next we have,

$$\mathrm{Curl}(\phi U) = \left\{ (\phi U^\alpha)_{;i} g^{ij} y^\beta_{;j} - (\phi U^\beta)_{;i} g^{ij} y^\alpha_{;j} \right\}$$

$$= \phi_{;i}\left[U^\alpha g^{ij} y^\beta_{;j} - U^\beta g^{ij} y^\alpha_{;j}\right] + \phi\left[U^\alpha_{;i} g^{ij} y^\beta_{;j} - U^\beta_{;i} g^{ij} y^\alpha_{;j}\right]$$

$$\therefore \mathrm{Curl}(\phi U) = U \times \nabla\phi + \phi\,\mathrm{Curl}\,U \qquad (2)$$

This agrees with the corresponding formula in $\mathcal{R}_n$.

$$\mathrm{Curl}\,\nabla\phi = (y^\alpha_{;j} g^{ji}\phi_{;i})_{;k} g^{kl} y^\beta_{;l} - (y^\beta_{;j} g^{ji}\phi_{;i})_{;k} g^{kl} y^\alpha_{;l}$$

$$= (y^\alpha_{;jk} g^{ji}\phi_{;i} + y^\alpha_{;j} g^{ji}\phi_{;ik}) g^{lk} y^\beta_{;l}$$

$$- (y^\beta_{;jk} g^{ji}\phi_{;i} + y^\beta_{;j} g^{ji}\phi_{;ik}) g^{kl} y^\alpha_{;l}$$

$$= \left(\Omega_{jk} g^{ji} g^{kl} y^\beta_{;l} N^\alpha \phi_{;i} - \Omega_{jk} g^{ji} g^{kl} y^\alpha_{;l} N^\beta \phi_{;i}\right)$$

$$= \phi_{;i}\,\Omega_{jk} g^{ji} g^{kl}\left(N^\alpha y^\beta_{;l} - N^\beta y^\alpha_{;l}\right)$$

$$= a^l (N^\alpha y^\beta_{;l} - N^\beta y^\alpha_{;l}),$$

where $a^l = g^{kl}\phi_{;i}\,\Omega_{jk} g^{ji}$

writing $a^l = A\varepsilon^l$ where ε^l = unit vector, and

$$A^2 = g_{lm} a^l a^m = g^{mj}\phi_{;i} g^{ki}\Omega_{km}\phi_{;j} g^{lj}\Omega_{lj}$$

$\therefore$ $\text{Curl}\,\nabla\phi = A\,(N^{\alpha} y^{\beta}_{;l}\,\mathcal{E}^{l} - N^{\beta} y^{\alpha}_{;l}\,\mathcal{E}^{l})$

observing that $\mathcal{N}^{\alpha}$ and $y^{\beta}_{;l}\,\mathcal{E}^{l}$ are orthogonal unit vectors we have using the result (1),

$(\text{magnitude})^2$ of $\text{curl}\,\nabla\phi = g^{m\beta} g^{ki} g^{lj}\,\phi_{;i}\phi_{;j}\,\Omega_{l\beta}\,\Omega_{mk}$

on the other hand in V_n $\text{Curl}\,\nabla\phi = 0$.

2. Let us denote the tensor $N \times E = N^{\alpha}E^{\beta} - N^{\beta}E^{\alpha}$ by $B^{\alpha\beta}$.

Consider $B.\text{curl}E = \frac{1}{2} a_{\alpha\gamma}\, a_{\beta\delta}\,[N^{\alpha}E^{\beta} - N^{\beta}E^{\alpha}]\,[E^{\gamma}_{;i} g^{ij} y^{\delta}_{;j} - E^{\delta}_{;i} g^{ij} y^{\gamma}_{;j}]$

$= a_{\alpha\gamma}\, a_{\beta\delta}\, N^{\alpha} E^{\beta} E^{\gamma}_{;i}\, g^{ij} y^{\delta}_{;j}$

$= (a_{\beta\delta}\, E^{\beta} y^{\delta}_{;j})(a_{\alpha\gamma}\, N^{\alpha} E^{\gamma}_{;i})\, g^{ij}$

$= (a_{\beta\delta}\, y^{\beta}_{;l}\, e^{l}\, y^{\delta}_{;j})(a_{\alpha\gamma}\, N^{\alpha} E^{\gamma}_{;i})\, g^{ij}$

$B.\text{curl}E = g_{lj}\, g^{ij}\, a_{\alpha\gamma}\, N^{\alpha} E^{\gamma}_{;l}\, e^{l} = a_{\alpha\gamma}\, N^{\alpha} E^{\gamma}_{;l}\, e^{l}$.

But $a_{\alpha\gamma} N^{\alpha} E^{\gamma} = 0$. Differentiating intrinsically in the direction of E we have,

$$a_{\alpha\gamma}\, N^{\alpha}_{;l}\, e^{l} E^{\gamma} + a_{\alpha\gamma}\, N^{\alpha} E^{\gamma}_{;l}\, e^{l} = 0$$

$\therefore$ $a_{\alpha\gamma}\, N^{\alpha} E^{\gamma}_{;l}\, e^{l} = -\, a_{\alpha\gamma}\, N^{\alpha}_{;l}\, e^{l} E^{\gamma}$ which reduces to $\frac{1}{R}$,

the normal curvature on substituting for $N^{\alpha}_{;l}$ by its value

2. Let us denote the tensor $N \times E = N^{\alpha}E^{\beta} - N^{\beta}E^{\alpha}$ * by B^{γ}

Consider B.curlE $= \frac{1}{2} a_{\alpha\gamma} a_{\beta\delta} \left[N^{\alpha}E^{\beta} - N^{\beta}E^{\alpha}\right]\left[E^{\gamma}_{;i} g^{ij} y^{\delta}_{;j} - E^{\delta}_{;i} g^{ij} y^{\gamma}_{;j}\right]$

$$= a_{\alpha\gamma} a_{\beta\delta} N^{\alpha}E^{\beta}E^{\gamma}_{;i} g^{ij} y^{\delta}_{;j}$$

$$= \left(a_{\beta\delta} E^{\beta} y^{\delta}_{;j}\right)\left(a_{\alpha\gamma} N^{\alpha}E^{\gamma}_{;i}\right) g^{ij}$$

$$= \left(a_{\beta\delta} y^{\beta}_{;l} e^{l} y^{\delta}_{;j}\right)\left(a_{\alpha\gamma} N^{\alpha}E^{\gamma}_{;i}\right) g^{ij}$$

$$B.\mathrm{curl}E = g_{lj} g^{ij} a_{\alpha\gamma} N^{\alpha}E^{\gamma}_{;l} e^{l} = a_{\alpha\gamma} N^{\alpha}E^{\gamma}_{;l} e^{l}.$$

But $a_{\alpha\gamma} N^{\alpha}E^{\gamma} = 0$. Differentiating intrinsically in the direction of E we have,

$$a_{\alpha\gamma} N^{\alpha}_{;l} e^{l} E^{\gamma} + a_{\alpha\gamma} N^{\alpha}E^{\gamma}_{;l} e^{l} = 0$$

$\therefore a_{\alpha\gamma} N^{\alpha}E^{\gamma}_{;l} e^{l} = - a_{\alpha\gamma} N^{\alpha}_{;l} e^{l} E^{\gamma}$ which reduces to $\frac{1}{R}$,

the normal curvature on substituting for $N^{\alpha}_{;l}$ by its value

$$- g^{lk} \Omega_{kl} y^{\alpha}_{;l}$$

$$\therefore (N \times E).\mathrm{curl}E = \frac{1}{R} \quad \ldots\ldots\ldots \quad (4)$$

Compare this with Weatherburn's result,

$$b.\mathrm{curl}\,t = -k_n^{*} \quad [12].$$

If C is an asymptotic line, B.curl$E = 0$

we shall show that Curl$N$ is a tensor of zero magnitude.

* Vide foot note to § 1, to account for the change in sign.

(Magnitude)^2 of curl N

$$= \tfrac{1}{2} a_{\alpha\gamma} a_{\beta\delta} \left[N^{\alpha}_{;l} g^{lk} y^{\beta}_{;k} - N^{\beta}_{;l} g^{lk} y^{\alpha}_{;k}\right]\left[N^{\gamma}_{;i} g^{ij} y^{\delta}_{;j} - N^{\delta}_{;i} g^{ij} y^{\gamma}_{;j}\right]$$

$$= \left(a_{\alpha\gamma} y^{\alpha}_{;k} y^{\gamma}_{;j}\right)\left(a_{\beta\delta} N^{\beta}_{;l} N^{\delta}_{;i}\right) g^{lk} g^{ij}$$
$$- \left(a_{\alpha\gamma} N^{\gamma}_{;i} y^{\alpha}_{;k}\right)\left(a_{\beta\delta} N^{\beta}_{;l} y^{\delta}_{;j}\right) g^{lk} g^{ij}$$

$$= \left(g_{kj} g^{ij}\right) g^{lk} \left(a_{\beta\delta} N^{\beta}_{;l} N^{\delta}_{;i}\right) - \left(a_{\alpha\gamma} N^{\gamma}_{;i} y^{\alpha}_{;k}\right)\left(a_{\beta\delta} N^{\beta}_{;l} y^{\delta}_{;j}\right) g^{lk} g^{ij}$$

$$= g^{lk}\left(a_{\beta\delta} N^{\beta}_{;l} N^{\delta}_{;k}\right) - \left(a_{\alpha\gamma} N^{\gamma}_{;i} y^{\alpha}_{;k}\right)\left(a_{\beta\delta} N^{\beta}_{;l} y^{\delta}_{;j}\right) g^{lk} g^{ij}$$

But $$N^{\beta}_{;l} = - g^{pq} \Omega_{pl} y^{\beta}_{;q}$$

and $$a_{\alpha\gamma} y^{\alpha}_{;k} N^{\gamma}_{;i} = - \Omega_{ki}$$

(Magnitude)^2 of curl $N = g^{lk} \Omega_{ki} \Omega_{lj} g^{ij} - g^{lk} \Omega_{ki} \Omega_{lj} g^{ij} = 0$

But curl N in V_{n+1} is not necessarily zero, since it is well known that Curl $N = 0$ is the necessary and sufficient condition for a system of geodesic parallels. The present result can howeder be compared to the result curl $n = 0$ for the unit normal vector on an ordinary surface [13].

It must also be noted that the magnitude of a tensor may be zero, without the individual components being zero. This will

$$-g\,(a_{\beta\delta} N^{\beta}_{;\ell} N^{\delta}_{;k}) - (a_{\alpha\gamma} N^{\gamma}_{;i} y^{\alpha}_{;k})(a_{\beta\delta} N^{\beta}_{;\ell} y^{\delta}_{;\gamma}) g^{\ell k} g^{i\gamma}$$

But $N^{\beta}_{;\ell} = -g^{pq}\Omega_{p\ell}\, y^{\beta}_{;q}$

and $a_{\alpha\gamma}\, y^{\alpha}_{;k} N^{\gamma}_{;i} = -\Omega_{ki}$

(Magnitude)^2 of curl $N = g^{\ell k}\Omega_{ki}\Omega_{\ell\gamma} g^{i\gamma} - g^{\ell k}\Omega_{ki}\Omega_{\ell\gamma} g^{i\gamma} = 0$

But curl N in V_{n+1} is not necessarily zero, since it is well known that Curl $N = 0$ is the necessary and sufficient condition for a system of geodesic parallels. The present result can however be compared to the result curl $n = 0$ for the unit normal vector on an ordinary surface [13].

It must also be noted that the magnitude of a tensor may be zero, without the individual components being zero. This will be clear from the definition of the magnitude of a tensor.

Next, $E\cdot \mathrm{Curl}\, N = E\cdot\left(N^{\alpha}_{;k} g^{\ell k} y^{\beta}_{;\ell} - N^{\beta}_{;k} g^{\ell k} y^{\alpha}_{;\ell}\right)$

$$= a_{\alpha\gamma} E^{\gamma}\left(N^{\alpha}_{;k} g^{\ell k} y^{\beta}_{;\ell} - N^{\beta}_{;k} g^{\ell k} y^{\alpha}_{;\ell}\right)$$

$$= a_{\alpha\gamma}\, y^{\gamma}_{;\delta} N^{\alpha}_{;k} e^{\delta} g^{\ell k} y^{\beta}_{;\ell} - a_{\alpha\gamma}\, y^{\gamma}_{;\delta}\, y^{\alpha}_{;\ell}\, e^{\delta} N^{\beta}_{;k} g^{\ell k}$$

$$= -\Omega_{\delta k} g^{\ell k} y^{\beta}_{;\ell} e^{\delta} - N^{\beta}_{;\delta} e^{\delta} = N^{\beta}_{;\delta} e^{\delta} - N^{\beta}_{;\delta} e^{\delta} = 0$$

$\therefore \quad E \cdot \operatorname{curl} N = 0$ . . . . (6)

These two results, viz. mag. $\operatorname{Curl} N = 0$ and $E \cdot \operatorname{Curl} N = 0$ correspond to the well known formula $\operatorname{Curl} \nabla \phi = 0$, where the ∇ and curl operations are confined to a specific V_n only.

Consider $N \cdot \operatorname{Curl} E = g_{\alpha\gamma} N^{\alpha} [E^{\gamma}_{;l} g^{lk} y^{\beta}_{;k} - E^{\beta}_{;l} g^{lk} y^{\gamma}_{;k}]$

$$= g_{\alpha\gamma} N^{\alpha} E^{\gamma}_{;l} g^{lk} y^{\beta}_{;k}$$

$$= - g_{\alpha\gamma} N^{\gamma}_{;l} E^{\alpha} g^{lk} y^{\beta}_{;k}$$

$$= g_{\alpha\gamma} g^{sm} \Omega_{sl} y^{\gamma}_{;m} y^{\alpha}_{;i} e^{i} y^{\beta}_{;k} g^{lk}$$

$$= (g_{\alpha\gamma} y^{\alpha}_{;i} y^{\gamma}_{;m}) g^{sm} \Omega_{sl} y^{\beta}_{;k} g^{lk} e^{i}$$

$$= g_{mi} g^{sm} \Omega_{sl} y^{\beta}_{;k} g^{lk} e^{i}$$

$$= \Omega_{sl} y^{\beta}_{;k} g^{lk} e^{s}$$

$\therefore \; N \cdot \operatorname{Curl} E = - N^{\beta}_{;k} e^{k} = - E \cdot \nabla N$

$= -$ (the intrinsic derivative of N) . . . . (7)

Magnitude of $N \cdot \operatorname{Curl} E$ = Magnitude of $N^{\beta}_{;k} e^{k}$

$$= g_{\alpha\beta} N^{\alpha}_{;i} e^{i} N^{\beta}_{;k} e^{k} = \tau_1^2 + \tau_2^2$$

$$= g^{lk} \Omega_{ki} \Omega_{lj} e^{i} e^{j} \; [14].$$

$$= - g_{\alpha\gamma} N^{\gamma}_{;l} E^{\alpha} g^{lk} y^{\beta}_{;k}$$

$$= g_{\alpha\gamma} g^{\delta m} \Omega_{\delta l} y^{\gamma}_{;m} y^{\alpha}_{;i} e^{i} y^{\beta}_{;k} g^{lk}$$

$$= (g_{\alpha\gamma} y^{\alpha}_{;i} y^{\gamma}_{;m}) g^{\delta m} \Omega_{\delta l} y^{\beta}_{;k} g^{lk} e^{i}$$

$$= g_{mi} g^{\delta m} \Omega_{\delta l} y^{\beta}_{;k} g^{lk} e^{i}$$

$$= \Omega_{\delta l} y^{\beta}_{;k} g^{lk} e^{\delta}$$

$$\therefore \underline{N \cdot \operatorname{Curl} E = - N^{\beta}_{;k} e^{k} = - E \cdot \nabla N}$$

$= -$ (the intrinsic derivative of N) . . . (7)

Magnitude of $N \cdot \operatorname{Curl} E$ = Magnitude of $N^{\beta}_{;k} e^{k}$

$$= g_{\alpha\beta} N^{\alpha}_{;i} e^{i} N^{\beta}_{;k} e^{k} = \tau_1^2 + \tau_2^2$$

$$= g^{lk} \Omega_{ki} \Omega_{lj} e^{i} e^{j} \quad [14].$$

From (7) we deduce that for an asymptotic line

$$\underline{E \cdot (N \cdot \operatorname{Curl} E) = 0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

while from (4) $\underline{(N \times E) \cdot \operatorname{Curl} E = 0}$

For a line of curvature, $N \cdot \operatorname{Curl} E$ is codirectional with E . . . . . . (9)

Also, $(N \cdot \operatorname{Curl} E) \cdot N = 0$ for any direction of V_n . . . . (10)

3. By an extension of the definition of the divergence of a vector, the divergence in V_{n+1} of a tensor T in V_{n+1} is defined by

$$\operatorname{div} T = T^{\alpha \cdots \cdots \beta}_{\gamma \cdots \cdots \delta, \beta} \quad [15]$$

We shall now define the divergence in V_n of a tensor T of order of contravariancy $p \geq 1$ in V_{n+1} by

$\nabla . T = g_{\alpha\delta}\, y^{\delta}_{;l}\, g^{lk}\, T^{\alpha \cdots}_{;k}$ which is a tensor of order of contravariancy $(p-1)$ . . . . (11)

Therefore, the divergence of a tensor of the second order is a vector and that of the vector is a scalar.

If T is a tensor of V_{n+1}, we shall define the tendency of T in V_n in any direction E by the expression $E \cdot \nabla T \cdot E$.

We have, $E \cdot \nabla T \cdot E = a_{\gamma\delta}\, y^{\delta}_{;l}\, e^{l}\, y^{\gamma}_{;c}\, g^{i\gamma}\, T^{\alpha \cdots}_{;\gamma} \cdots a_{\alpha\beta}\, y^{\beta}_{;l}\, e^{l}$

$$= (g_{il}\, g^{i\gamma}) e^{l}\, a_{\alpha\beta}\, T^{\alpha \cdots}_{;\gamma} \cdots y^{\beta}_{;l}\, e^{l}$$

$$E \cdot \nabla T \cdot E = a_{\alpha\beta}\, T^{\alpha \cdots \cdots}_{;\gamma}\, e^{\gamma}\, y^{\beta}_{;l}\, e^{l} \quad . . . . (12)$$

But the right side = tendency of T w.r.t. C in V_{n+1}

Therefore, the tendency in V_{n+1} of a tensor T in V_{n+1} is order of contravariancy $p-1)$ . . . . (11)

Therefore, the divergence of a tensor of the second order is a vector and that of the vector is a scalar.

If T is a tensor of V_{n+1}, we shall define the tendency of T in V_n in any direction E by the expression $E.\nabla T.E$.

We have, $E.\nabla T.E = a_{\gamma\delta}\, y^{\delta}_{;l}\, e^l\, y^{\gamma}_{;i}\, g^{ij}\, T^{\alpha\cdots}_{;j}\; a_{\alpha\beta}\, y^{\beta}_{;l}\, e^l$

$$= (g_{il}\, g^{ij})\, e^l\, a_{\alpha\beta}\, T^{\alpha\cdots}_{;j}\; y^{\beta}_{;l}\, e^l$$

$$E.\nabla T.E = a_{\alpha\beta}\, T^{\alpha\cdots}_{;j}\; e^j\, y^{\beta}_{;l}\, e^l \quad \ldots\ldots \quad (12)$$

But the right side $=$ tendency of T w.r.t. C in V_{n+1}

Therefore, the tendency in V_{n+1} of a tensor T in V_{n+1} is equal to its tendency in V_n.

If u^i is a vector of V_n, with components $U^{\alpha} = y^{\alpha}_{;i}\, u^i$ in V_{n+1}, $E.\nabla U.E = a_{\alpha\beta}\, U^{\alpha}_{;j}\, e^j\, y^{\beta}_{;l}\, e^l$

$$= a_{\alpha\beta}\, (y^{\alpha}_{;l}\, u^l)_{;j}\, e^j\, y^{\beta}_{;l}\, e^l$$

$$= a_{\alpha\beta}\, y^{\alpha}_{;l}\, u^l_{;j}\, e^j\, y^{\beta}_{;l}\, e^l = g_{lk}\, e^l_{;j}\, e^j\, e^l$$

$=$ tendency of u w.r.t. C in V_n.

Hence, the tendency of U in V_{n+1} is equal to the tendency of U in V_n and when U belongs to V_n, these are both equal to the tendency of u in V_n . . . . (14)

In particular, the tendency of the normal vector

$$= E \cdot \nabla N \cdot E = -\frac{1}{R} = \text{- normal curvature.}$$

Theorem: The sum of the tendencies of any tensor T of V_{n+1} in n mutually orthogonal directions of an ennuple of V_n is equal to the divergence in V_n of the tensor T .

Proof:- The tendency of T w.r.t. $e_{h|} = a_{\alpha\beta} T^{\alpha \cdots}_{;\gamma} e_{h|}^{\gamma} y^{\beta}_{;\delta} e_{h|}^{\delta}$
$(h = 1, 2, \ldots n)$

$\therefore$ Sum of the tendencies of T w.r.t. $e_{h|}$

$$= \sum_h a_{\alpha\beta} T^{\alpha \cdots}_{;\gamma} e_{h|}^{\gamma} y^{\beta}_{;\delta} e_{h|}^{\delta}$$

$$= \sum_h e_{h|}^{\gamma} e_{h|}^{\delta} a_{\alpha\beta} T^{\alpha \cdots}_{;\gamma} y^{\beta}_{;\delta}$$

$$= g^{\gamma\delta} a_{\alpha\beta} y^{\beta}_{;\delta} T^{\alpha \cdots}_{;\gamma} = \operatorname{div} T, \qquad \text{by definition.}$$

Cor.: Sum of the tendencies of the unit normal vector in V_{n+1} w.r.t. an orthogonal ennuple of directions of V_n = - mean curvature of V_n .

This may be compared with the corresponding result known

Proof:- The tendency of T w.r.t. $e_{h|} = g_{\alpha\beta} T_{;\gamma} \; e_{h|}^{\gamma} y^{\beta}_{;\delta} e_{h|}^{\delta}$
$(h = 1, 2, \ldots n)$

Sum of the tendencies of T w.r.t. $e_{h|}$

$$= \sum_h g_{\alpha\beta} T^{\alpha \cdots}_{;\gamma} \; e_{h|}^{\gamma} y^{\beta}_{;\delta} e_{h|}^{\delta}$$

$$= \sum_h e_{h|}^{\gamma} e_{h|}^{\delta} g_{\alpha\beta} T^{\alpha \cdots}_{;\gamma} \; y^{\beta}_{;\delta}$$

$$= g^{\gamma\delta} g_{\alpha\beta} y^{\beta}_{;\delta} T^{\alpha \cdots}_{;\gamma} = \operatorname{div} T,$$ by definition.

Cor.: Sum of the tendencies of the unit normal vector in V_{n+1} w.r.t. an orthogonal ennuple of directions of V_n = - mean curvature of V_n.

This may be compared with the corresponding result known for a vector in V_n [16].

4. In ordinary space, we have the well known Green's identity
$\operatorname{div}(a \operatorname{grad} b - b \operatorname{grad} a) = a \operatorname{div} \operatorname{grad} b - b \operatorname{div} \operatorname{grad} a.$
It is quite easy to extend this ~~fxxxk~~ to a Riemannian V_n also. We shall now extend this for the operations div and grad defined in this paper, for tensors of V_{n+1} the operations being in the field of V_n.

If ϕ and ψ are two scalar functions of $\mathcal{V}_n$, we have

$$\operatorname{div}(\phi \operatorname{grad}\psi - \psi \operatorname{grad}\phi) = \operatorname{div}(\phi\, y^{\alpha}_{;l} g^{lk} \psi_{;k} - \psi\, y^{\alpha}_{;l} g^{lk} \phi_{;k})$$

$$= a_{\alpha\beta}\, y^{\beta}_{;i} g^{ij} (\phi\, y^{\alpha}_{;l} g^{lk} \psi_{;k} - \psi\, y^{\alpha}_{;l} g^{lk} \phi_{;k})_{;j}$$

$$= \phi\, \psi_{;kj} g^{kj} - \psi\, \phi_{;kj} g^{kj}$$

Also, $\phi \operatorname{div}\operatorname{grad}\psi - \psi \operatorname{div}\operatorname{grad}\phi$

$$= \phi\, a_{\alpha\beta}\, y^{\beta}_{;i} g^{ij} (y^{\alpha}_{;l} g^{lk} \psi_{;k})_{;j} - \psi\, a_{\alpha\beta}\, y^{\beta}_{;i} g^{ij} (y^{\alpha}_{;l} g^{lk} \phi_{;k})_{;j}$$

$$= \phi\, \psi_{;kj} g^{kj} - \psi\, \phi_{;kj} g^{kj}$$

$$\therefore \operatorname{div}(\phi \operatorname{grad}\psi - \psi \operatorname{grad}\phi) = \phi \operatorname{div}\operatorname{grad}\psi - \psi \operatorname{div}\operatorname{grad}\phi. \quad \ldots \quad (15)$$

Next, $\operatorname{div}(U \times V) = a_{\alpha\gamma}\, y^{\gamma}_{;i} g^{ij} (U^{\alpha} V^{\beta} - U^{\beta} V^{\alpha})_{;j}$

$$= U.\operatorname{curl} V - V.\operatorname{curl} U \quad \ldots \quad (16)$$

$$\nabla(U.V) = y^{\beta}_{;i} g^{ij} (a_{\alpha\gamma} U^{\alpha} V^{\gamma})_{;j}$$

$$= V.\nabla U + U.\nabla V + V.\operatorname{curl} U + U.\operatorname{curl} V \quad \ldots \quad (17)$$

(16) and (17) are known results when U and V and ∇ are all defined in $\mathcal{V}_n$. [17]

We can obviously extend the results (16) and (17) for any two tensors U and V of $\mathcal{V}_n$, ∇ being the operator defined

$$= \phi\, a_{\alpha\beta}\, y^{\beta}_{;i}\, g^{i\gamma}\left(y^{\alpha}_{;l}\, g^{lk}\, \psi_{;k}\right)_{;\gamma} - \psi\, a_{\alpha\beta}\, y^{\beta}_{;i}\, g^{i\gamma}\left(y^{\alpha}_{;l}\, g^{lk}\, \phi_{;k}\right)_{;\gamma}$$

$$= \phi\, \psi_{;k\gamma}\, g^{k\gamma} - \psi\, \phi_{;k\gamma}\, g^{k\gamma}$$

$$\therefore \operatorname{div}(\phi \operatorname{grad}\psi - \psi \operatorname{grad}\phi) = \phi \operatorname{div}\operatorname{grad}\psi - \psi \operatorname{div}\operatorname{grad}\phi . \quad \ldots \ldots (15)$$

Next, $\operatorname{div}(U \times V) = a_{\alpha\gamma}\, y^{\gamma}_{;i}\, g^{i\gamma}\left(U^{\alpha}V^{\beta} - U^{\beta}V^{\alpha}\right)_{;\gamma}$

$$= U.\operatorname{curl} V - V.\operatorname{curl} U \quad \ldots \ldots \ldots (16)$$

$$\nabla(U.V) = y^{\beta}_{;i}\, g^{i\gamma}\left(a_{\alpha\gamma} U^{\alpha} V^{\gamma}\right)_{;\gamma}.$$

$$= V.\nabla U + U.\nabla V + V.\operatorname{curl} U + U.\operatorname{curl} V \quad \ldots (17)$$

(16) and (17) are known results when U and V and ∇ are all defined in V_n. [17]

We can obviously extend the results (16) and (17) for any two tensors U and V of V_{n+1}, ∇ being the operator defined in this paper, operating in V_n on functions of V_{n+1}.

Consider $\operatorname{div}(N \times E)$.

$$\operatorname{div}(N \times E) = a_{\alpha\gamma}\, y^{\gamma}_{;l}\, g^{lk}\left(N^{\alpha}E^{\beta} - N^{\beta}E^{\alpha}\right)_{;k}$$

$$= a_{\alpha\gamma}\, y^{\gamma}_{;l}\, g^{lk}\left[N^{\alpha}_{;k}E^{\beta} + N^{\alpha}E^{\beta}_{;k} - N^{\beta}_{;k}E^{\alpha} - N^{\beta}E^{\alpha}_{;k}\right].$$

$$\operatorname{div}(N \times E) = -\Omega_{lk}\, g^{lk}\, E^{\beta} - N^{\beta}_{;k}\, e^{k} - N^{\beta} \operatorname{div} E \quad \ldots \ldots (18)$$

$$= - ME^{\beta} - N^{\beta}_{,k} e^{k} - N^{\beta} \operatorname{div} e,$$

Since $\operatorname{div} E = \operatorname{div} e$ in V_n ·

~~since~~

in virtue of equation (3), Section I, of this paper.

From (16), $\operatorname{div}(N \times E) = (-E \cdot \operatorname{curl} N + N \cdot \operatorname{curl} E)$ . . . (19)

from (18) and (19) we have in virtue of (7)

$ME + N \operatorname{div} e = E \cdot \operatorname{curl} N$ . . . . (20)

$N \cdot E \cdot \operatorname{curl} N = \operatorname{div} e$ . . . . (21)

and $(\text{magnitude})^2$ of $E \cdot \operatorname{curl} N = M^2 + (\operatorname{div} e)^2$, M being the mean curvature of V_n . . . . (22)

From (18) $(\text{magnitude})^2$ of $\operatorname{div}(N \times E) = M^2 + (\tau_1^2 + \tau_2^2) + (\operatorname{div} e)^2 + 2M\tau$ . . . . (23)

where τ_1 and τ_2 are respectively the first and second curvatures in V_{n+1} of the geodesic tangent to C in V_n .

Taking scalar products with N on either side of (18) we have $N \cdot \operatorname{div}(N \times E) = -\operatorname{div} e = -N \cdot E \cdot \operatorname{curl} N$ . . . . . (24)

in virtue of the properties $N \cdot E = 0$ and $(N \cdot \operatorname{curl} E) \cdot N = 0$

Taking scalar products with E on either side of (18) and (20)

$$ME + N \operatorname{div} e = E \cdot \operatorname{curl} N \quad \ldots \ldots \quad (20)$$

$$N \cdot E \cdot \operatorname{curl} N = \operatorname{div} e \quad \ldots \ldots \quad (21)$$

and (magnitude)^2 of $E \cdot \operatorname{curl} N = M^2 + (\operatorname{div} e)^2$, M being the mean curvature of $\mathcal{V}_n$ . . . . (22)

From (18) (magnitude)^2 of $\operatorname{div}(N \times E) = M^2 + (\tau_1^2 + \tau_2^2) + (\operatorname{div} e)^2 + 2M\tau_1$ . . . (23)

where τ_1 and τ_2 are respectively the first and second curvatures in $\mathcal{V}_{\lambda+1}$ of the geodesic tangent to C in $\mathcal{V}_n$.

Taking scalar products with N on either side of (18) we have $N \cdot \operatorname{div}(N \times E) = -\operatorname{div} e = -N \cdot E \cdot \operatorname{curl} N$ . . . . . (24)

in virtue of the properties $N \cdot E = 0$ and $N \cdot \operatorname{curl}(E) \cdot N = 0$

Taking scalar products with E on either side of (18) and (20) respectively we have for any direction along a hypersurface,

$E \cdot \operatorname{div}(N \times E) = -M + \frac{1}{R}$, where $\frac{1}{R}$ = normal curvature . . (2

and $E \cdot \operatorname{curl} N \cdot E = M =$ mean curvature of $\mathcal{V}_n$ 2

If $e_{h|}$ is a geodesic congruence then

$$\operatorname{div} e_{h|} = \sum_l \gamma_{hll} \quad [18]$$

$$= 0 \quad \text{since } \gamma_{hll} = 0 \quad \text{for } l = 1, 2, \ldots, n$$

$\therefore N \cdot \operatorname{div}(N \times E) = 0$ and $N \cdot E \cdot \operatorname{Curl} N = 0$

For a geodesic congruence the unit normal vector is orthogonal to $\operatorname{div}(N \times E)$ as well as to $E \cdot \operatorname{Curl} N$.

References

(R.G. = Riemannian geometry; D.G. = Differential geometry)

1 C.E.Weatherburn. R.G.: Ch.VIII, p.136, equation (40).

2 L.P.Eisenhart R.G.: p.219, Ex.9.

3. Paper III of this thesis, p.31, Theorem 6.

4 C.E.Weatherburn. loc.cit. Ch.IX, p.152, § 84.

5 and 7 Do. D.G. Vol.II, Ch.VIII, § 86.

6 Do. R.G. Ch.VIII, § 76.

8 Do. D.G. Vol.II, Ch.VIII, § 86, expressions (36) & (37).

9 Paper III of this Thesis, p. 29.

10 E.H.Neville. Multilinear functions of direction, p.58, th^m 6.231.

11 C.E.Weatherburn. R.G. § 35, p.60, expression (13).

12 Do. D.G. Vol.II, § 8, p.18.

13 Do. D.G. Vol.I, § 118, p.229.

14 Paper III of this Thesis: p. 30, equation (4.2)

15 T.Y.Thomas. Differential invariants of generalized spaces. p.48, 3°.

16 C.E.Weatherburn. R.G. Chap.VIII, p.131, equation (23).

17 Do. D.G. Vol.II, ex.11, p.69.

18 Do. R.G. Ch.VI, p.109, ex.1.

B i b l i o g r a p h y

1 Allendoerfer: Riemannian Geometry. Thesis, 1951.

2 Blaschke. W. Frenet's Formulen für den Raum von Riemann. Math. Zeitschrift 6 (1920), 94-99.

3 Bornpiani, E. Spazi Riemanniani luoghi di varieta totalmente geodetiche. Rend di. Palemo. 48 (1924). pp. 121-134.

4 Campbell, J.E. Course of Differential Geometry, 1926.

5 Carban, E. Lecons Surla Geometrie des espaces de Riemann 1928

6 Cutler, E.H. - On the curvature of a curve in a Riemannian space Transactions of the American Mathematical Society. 33 (1931), 832-838.

7 Darboux, G. Lecons Surla theorie generale Des Surfaces, Vol.II, 1915.

8 Duchek, A -- Mayer, W.: Lehrbuch der Differential Geometrie. Band II. Riemannsche Geometrie. 1930.

9 Edmund Wright, J; Invariants of Quadratic Differential forms. 1908.

10 Eisenhart L.P. Differential Geometry 1940.

11 Do. Riemannian Geometry, 1948.

12 Forsyth, A.R. Differential Geometry.

13 Graustein W.C. The Geometry of Riemann spaces. Transactions of the American Mathematical Society. 36, pp.542-585.

14 Hayden H.A. Asymptotic lines in a V_m in V_n Proceedings of the London Mathematical Society (2) 33 (1930) pp.22-31.

15 Manikarnikamma S.N. The Second Curvature of a geodesic on a hypersurface of a Riemannian space, Proc. Indian Academy of Sciences. 36 (1952

5 Carban, E. Lecons Surla Geometrie des espaces de Riemann 192

6 Cutler, E.H. On the curvature of a curve in a Riemannian space Transactions of the American Mathematical Society. 33 (1931), 832-838.

7 Darboux, G. Lecons Surla theorie generale Des Surfaces, Vol.II, 1915.

8 Duchek, A -- Mayer, W.: Lehrbuch der Differential Geometrie. Band II. Riemannsche Geometrie. 1930.

9 Edmund Wright, J; Invariants of Quadratic Differential forms. 1908.

10 Eisenhart L.P. Differential Geometry 1940.

11 Do. Riemannian Geometry, 1948.

12 Forsyth, A.R. Differential Geometry.

13 Graustein W.C. The Geometry of Riemann spaces. Transactions of the American Mathematical Society. 36, pp.542-585.

14 Hayden H.A. Asymptotic lines in a V_m in V_n Proceedings of the London Mathematical Society (2) 33 (1930) pp.22-31.

15 Manikarnikamma S.N. The Second Curvature of a geodesic on a hypersurface of a Riemannian space, Proc. Indian Academy of Sciences. 36 (1952) pp.175-181.

16 McConnell A.J. Absolute Differential Calculus, 1931.

17 Michal A.D. Matrix and Tensor Calculus.

18 Mishra R.S. Hyper - Asymptotic curves of a Riemannian - Hypersurface. Math. Student XX (1952), 63-65.

19 Neville E.H. Multilinear Functions of Direction. 1921.

20 Oswald Veblern: Invariants of Quadratic Differential Forms 1927.

21 Ram Behari: (a) Lectures on the Differential Geometry of Ruled surface.

21 (b) Equilateral Osculatintquadrics of Ruled Surfaces.
Jour. Indian Math. Soc. Vol.XX (1933).

(c) A note on Laguerre's Function. Jour. Indian Math. Soc.
New Series. Vol.III (1939)

22 Rangachariar V. On three Orthogonal Congruences of Curves.
Jour. Indian Math. Soc. Vol.II (New Series), 1936-37,
pp.216-219.

23 Ruse H.S.: (a) On Simply harmonic. Kappa Spaces of four
dimensions. Proc. Lond. Math. Soc. (2) 50,
1948, pp.317-329.

(b) Three-dimensional spaces of recurrent curvatures. Ibid. pp.438-446.

24 Schouten J.A. Der Ricci - Kalkiil. 1924.

25 Schouten J.A. and Struik D.J.: (a) Einfuhrung in die neueren
Methoden der Differential Geometrie, Bands
I and II.

(b) Über Krummungseigenschaften einer m-dimensionalen Mannigfaltigkeit, die in einer
n-dimensionalen Mannigfaltigkeit mit
beliebiger quadratischer Massbestimmung
eingebettet ist.

26 Sokolnikoff I.S. Tensor Analysis. Theory and Applications
1951.

27 Struik D.J. Lectures on classical Differential Geometry
1950.

28 Synge J.L. Normals and curvatures of a curve in a
Riemannian Manifold. Prof. Int. Math. Congress
Toronto. I (1928), 857-862.

29 Synge J.L. and Schild A. Tensor Calculus 1949.

30 Thomas T.Y. Differential Invariants of Generalized spaces.
1934. (pp.45, 139).

31 Tuillio Levi-civita: The Absolute Differential Calculus.
1927.

32 Vranceanu G. Lecons de Geometrie Differentialle. Vol.I and
II, 1927.

33 Walker A.G. On Ruse's spaces of recurrent curvature.

24 Schouten J.A. Der Ricci - Kalkiil. 1924.

25 Schouten J.A. and Struik D.J.: (a) Einfuhrung in die neueren Methoden der Differential Geometrie, Bands I and II.

(b) Über Krummungseigenschaften einer m-dimensionalen Mannigfaltigkeit, die in einer n-dimensionalen Mannigfaltigkeit mit beliebiger quadratischer Massbestimmung eingebettet ist.

26 Sokolnikoff I.S. Tensor Analysis. Theory and Applications 1951.

27 Strukk D.J. Lectures on classical Differential Geometry 1950.

28 Synge J.L. Normals and curvatures of a curve in a Riemannian Manifold. Prof. Int. Math. Congress Toronto. I (1928), 857-862.

29 Synge J.L. and Schild A. Tensor Calculus 1949.

30 Thomas T.Y. Differential Invariants of Generalized spaces. 1934. (pp.45, 139).

31 Tuillio Levi-civita: The Absolute Differential Calculus. 1927.

32 Vranceanu G. Lecons de Geometrie Differentialle. Vol.I and II, 1927.

33 Walker A.G. On Ruse's spaces of recurrent curvature. Proc. Lond. Math. Soc. (2), Vol.52, 1951.

34 Weatherburn C.E. Differential Geometry, Vol.I and II. 1930.

35 Do. Riemannian Geometry. 1950.

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