PART III

ON AN IDENTITY OF RAMANUJAN CONNECTED WITH ELLIPTIC FUNCTIONS: ITS EXTENSION AND APPLICATION

1. In connection with the development of obtaining the expression for $\gamma_{2k}(n)$, Ramanujan [7, pp.138-139] has obtained an identity* which is equivalent to the following:

$$\left(\zeta(u)-\frac{\eta_1 u}{\omega_1}\right)^2-\wp(u)=\left(\frac{2\pi}{\omega_1}\right)^2\left[-\frac{1}{24}+\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\cos\frac{n\pi u}{\omega_1}\right] \quad (3.1)$$

(with the usual notations of Elliptic functions).

We give here a proof of (3.1) (via contour integration) and obtain the following new identity along the same lines:

$$\left(\frac{\omega_1}{2\pi}\right)^3\left(\zeta(u)-\frac{\eta_1 u}{\omega_1}\right)^3=\left(\frac{1}{4}\cot\frac{\pi u}{2\omega_1}\right)^3-\frac{3}{2}\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^3}\sin\frac{n\pi u}{\omega_1}+$$

$$+\frac{3}{4}\sum_{n=1}^{\infty}\frac{n q^{2n}}{(1-q^{2n})^2}\sin\frac{n\pi u}{\omega_1}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\sin\frac{n\pi u}{\omega_1}-$$

$$-\frac{1}{16}\sum_{n=1}^{\infty}\frac{(2n^2+1)\,q^{2n}}{1-q^{2n}}\sin\frac{n\pi u}{\omega_1}+\frac{3}{8}\cot\frac{\pi u}{2\omega_1}\sum_{n=1}^{\infty}\frac{n q^{2n}}{1-q^{2n}}$$

$$+\frac{3}{2}\left(\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi u}{\omega_1}\right)\left(\sum_{n=1}^{\infty}\frac{n q^{2n}}{1-q^{2n}}\right) \quad (3.2)$$

Proof of (3.1)

The identity (3.1) is equivalent to (putting $e^{i\pi u/\omega_1}=z$)

$$f^2(z)=\left(\frac{1+z}{1-z}\right)^2-8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}-8\sum_{n=1}^{\infty}\frac{q^n(z^n+z^{-n})}{(1-q^n)^2}+$$

(1)

$$+4\sum_{n=1}^{\infty}\frac{n q^n(z^n+z^{-n})}{1-q^n},$$

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* Identity No (17)

where

$$f(z) \equiv \frac{1+z}{1-z} + 2\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}(z^n - z^{-n})$$

$$= \frac{1+z}{1-z} + 2\sum_{n=1}^{\infty} \frac{q^n z}{1-q^n z} - 2\sum_{n=1}^{\infty} \frac{q^n z^{-1}}{1-q^n z^{-1}} .$$

We prove that the Laurent Expansion of $f^2(z)$ in $|q|<|z|<1$ is equal to the right side of (i). The function $f(z)$ satisfies the functional relation

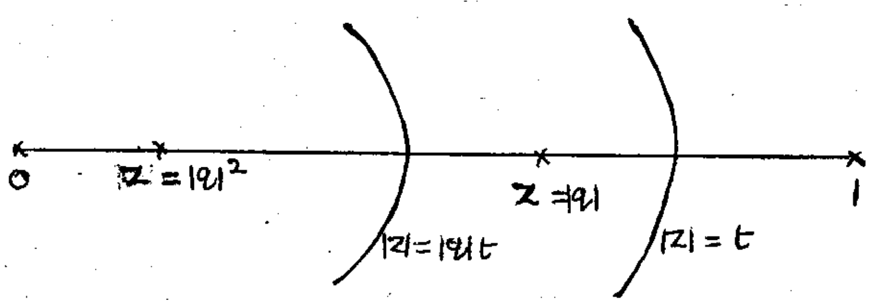
(ii) $$f(qz) = 2 + f(z) .$$

Let

$$f^2(z) = \sum_{-\infty}^{\infty} a_n z^n ,$$

be the Laurent series expansion of $f^2(z)$ in $|q|<|z|<1$ then

$$a_n = \frac{1}{2\pi i}\int_{|z|=t} \frac{f^2(z)}{z^{n+1}}\, dz .$$



Putting $z = u/q$ in the above integral for a_n and using the functional relation (ii) we obtain

$$\frac{a_n}{q^n} = \frac{1}{2\pi i}\int_{|u|=|q|t} \frac{(f(u)-2)^2}{u^{n+1}}\,du$$

$$= \frac{1}{2\pi i}\int_{|u|=t} \frac{(f(u)-2)^2}{u^{n+1}}\,du \quad - R_n ,$$

where R_n is the residue of $(f(u)-2)^2/u^{n+1}$ at $u=q$ at which the function $(f(u)-2)^2/u^{n+1}$ has a pole of second order.

Hence

$$\frac{a_n}{q^n} = \frac{1}{2\pi i}\int_{|u|=t} \frac{f^2(u)}{u^{n+1}}\,du \;-\; 4\cdot\frac{1}{2\pi i}\int_{|u|=t}\frac{f(u)}{u^{n+1}}\,du - R_n .$$

The first integral on the right side is a_n. The second integral on the right side is equal to the coefficient of u^n in the expansion of $f(u)$ and is equal to $2/1-q^n$.

Hence we have

$$\frac{a_n}{q^n} = a_n - 4\left(\frac{2}{1-q^n}\right) - R_n$$

or

(iii) $$a_n = -\frac{8q^n}{(1-q^n)^2} - R_n\,\frac{q^n}{1-q^n} .$$

To find R_n we proceed as follows :

$$f(z)-2 = \frac{3z-1}{1-z} + 2\sum_{n=1}^{\infty}\frac{q^n z}{1-q^n z} - 2\sum_{n=2}^{\infty}\frac{q^n}{z-q^n} - \frac{2q}{z-q} .$$

We put $z = q+t$ in this and write

$$\frac{3q+3t-1}{1-q-t} + 2\sum_{n=1}^{\infty}\frac{q^n(q+t)}{1-q^{n+1}-q^n t} - 2\sum_{n=2}^{\infty}\frac{q^n}{t+q-q^n}$$

$$= \psi(t) \equiv b_0 + b_1 t + b_2 t^2 + \cdots\cdots ,$$

then

$$f(q+t)-2 = \psi(t) - 2q/t .$$

Now R_n is the coefficient of $1/t$ in the expansion of

$$\left(\psi(t) - 2q/t\right)^2 \frac{(1+t/q)^{-\overline{n+1}}}{q^{n+1}}$$

or in the expansion of

$$\left(2q - b_0 t - b_1 t^2 - b_2 t^3 - \cdots\cdots\right)^2 \frac{(1+t/q)^{-(n+1)}}{q^{n+1}t^2} .$$

Hence

$$R_n = \frac{1}{q^{n+1}}\left[-(n+1)4q - 4qb_0\right]$$

i.e.

$$R_n q^n = -4(n+1) - 4b_0 .$$

b_0 (the constant term in $\psi(t)$)

$$= \frac{3q-1}{1-q} + 2\sum_{n=1}^{\infty} \frac{q^{n+1}}{1-q^{n+1}} - 2\sum_{n=2}^{\infty} \frac{q^{n-1}}{1-q^{n-1}} = -1 .$$

Thus we have

$$R_n = -4n/q^n .$$

Substituting this in (iii) we have

$$a_n = -\frac{8q^n}{(1-q^n)^2} + \frac{4n}{1-q^n}, \quad n \neq 0 .$$

Also, changing n to $-n$

$$a_{-n} = -\frac{8q^n}{(1-q^n)^2} + \frac{4nq^n}{1-q^n}, \quad n \neq 0 .$$

Hence we have proved

$$f^2(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^n$$

(iv)

$$= a_0 - 8\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}(z^n + z^{-n}) + 4\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}(z^n+z^{-n}) + \left(\frac{1+z}{1-z}\right)^2 - 1 .$$

$\left(\text{Since } 4\sum_{1}^{\infty} n z^n = \left(\frac{1+z}{1-z}\right)^2 - 1\right)$.

To determine a_0 we consider the limit of (iv) as $z \to 1$:

we have

$$\lim_{z\to 1}\left[f^2(z) - \left(\frac{1+z}{1-z}\right)^2\right]$$

$$= \lim_{z\to 1}\left[a_0 - 1 - 8\sum_{n=1}^{\infty} \frac{q^n(z^n+z^{-n})}{(1-q^n)^2} + 4\sum_{n=1}^{\infty} \frac{nq^n(z^n+z^{-n})}{1-q^n}\right].$$

The left side of the above relation tends to

$$-16\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\quad\left(=-16\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\right),$$

since

$$f(z) = \frac{1+z}{1-z} + 2\sum_{n=1}^{\infty}\frac{q^n(z^n-z^{-n})}{1-q^n},$$

We have

$$f^2(z) - \left(\frac{1+z}{1-z}\right)^2 = 4\left(\frac{1+z}{1-z}\right)\sum_{n=1}^{\infty}\frac{q^n(z^n-z^{-n})}{1-q^n} + 4\left[\sum_{n=1}^{\infty}\frac{q^n(z^n-z^{-n})}{(1-q^n)^2}\right]^2,$$

and the right side tends to

$$a_0 - 1 - 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

Hence

$$a_0 = 1 - 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\quad\left(=1-8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\right).$$

Substituting the value of a_0 in (iv) we have (i). This completes the proof of the Ramanujan identity (3.1).

Proof of (3.2)

The identity (3.2) is equivalent to

$$f^3(z) = \left(\frac{1+z}{1-z}\right)^3 + 48\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^3}(z^n - z^{-n}) - 24\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z^n - z^{-n}) -$$

(v) $$-24\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)^2}(z^n - z^{-n}) + \sum_{n=1}^{\infty}(4n^2+2)\frac{q^n}{1-q^n}(z^n - z^{-n}) -$$

$$-24\left(\frac{1+z}{1-z}\right)\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 48\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(z^n - z^{-n})\right)\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right),$$

where $f(z)$ is defined as before.

We prove that the Laurent expansion of $f^3(z)$ in $|q| < |z| < 1$ is equal to the right side of (v).

Let

$$f^3(z) = \sum_{n=-\infty}^{\infty} a_n' z^n, \quad |q| < |z| < 1.$$

Then

$$a_n' = \frac{1}{2\pi i}\int_{|z|=t}\frac{f^3(z)}{z^{n+1}}\,dz.$$

By putting $z = u/q$ in this and using the functional relation (ii) we obtain

$$\frac{a_n'}{q^n} = \frac{1}{2\pi i}\int_{|u|=t}\frac{(f(u)-2)^3}{u^{n+1}} - R_n',$$

where R_n' is the residue of $(f(u)-2)^3/u^{n+1}$ at $u = q$ at which the function $(f(u)-2)^3/u^{n+1}$ has a pole of third order.

Hence

$$\frac{a_n'}{q^n} = \frac{1}{2\pi i}\int_{|u|=t} \frac{f^3(u)}{u^{n+1}}\,du - 6\cdot\frac{1}{2\pi i}\int_{|u|=t}\frac{f^2(u)}{u^{n+1}}\,du + 12\cdot\frac{1}{2\pi i}\int_{|u|=t}\frac{f(u)}{u^{n+1}}\,du - R_n'.$$

The first integral on the right side is a_n^1.

The second integral on the right side is the coefficient of u^n in the Laurent expansion of $f^2(u)$ and is equal to

$$a_n = -\frac{8q^n}{(1-q^n)^2} + \frac{4n}{1-q^n} \qquad \text{(proved above)}.$$

Hence we have

$$\frac{a_n'}{q^n} = a_n^1 - 6\left[-8\frac{q^n}{(1-q^n)^2} + \frac{4n}{1-q^n}\right] + 12\cdot\frac{2}{1-q^n} - R_n^1$$

$$a_n^1 = 48\frac{q^n}{(1-q^n)^3} - 24\frac{nq^n}{(1-q^n)^2} + 24\frac{q^n}{(1-q^n)^2} - \frac{R_n^1 q^n}{1-q^n}.$$

R_n^1 is the coefficient of $1/t$ in the expansion of

$$\left(\psi(t) - 2q/t\right)^3 \frac{(1+t/q)^{-\overline{n+1}}}{t^3 q^{n+1}}$$

or in the expansion of

$$-\left(2q - b_0 t - b_1 t^2 - b_2 t^3 - \dots\right)^3 (1+t/q)^{-\overline{n+1}}$$

or in

$$-\frac{1}{t^3 q^{n+1}}\left[8q^3 - 12q^2 b_0 t - (12q^2 b_1 - 6qb_0^2)t^2 + \dots\right]\times$$
$$\times\left[1 - \frac{(n+1)}{q}t + \frac{(n+1)(n+2)}{2q^2}t^2 - \dots\right].$$

Hence

$$R_n^1 = -\frac{1}{q^{n+1}}\left[4q(n+1)(n+2) + 12q(n+1)b_0 - 12q^2 b_1 + 6b_0^2 q\right],$$

where

$$b_0 \text{ (the constant term in } \psi(t)) = -1$$

and

$$b_1 \text{ (coefficient of } t \text{ in } \psi(t)) = \frac{d}{dt}\psi(t)\Big|_{t=0} =$$

$$\frac{2}{(1-q)^2} + 2\sum_{n=1}^{\infty}\frac{q^n}{(1-q^{n+1})^2} + 2\sum_{n=2}^{\infty}\frac{q^{n-2}}{(1-q^{n-1})^2} = 4\sum_{n=0}^{\infty}\frac{q^n}{(1-q^{n+1})^2}.$$

With these values of b_0 and b_1 we have

$$R_n^1 = -\frac{1}{q^n}\left[4n^2+2 - 48\sum_{m=1}^{\infty}\frac{q^m}{(1-q^m)^2}\right].$$

Hence for $n \neq 0$

$$a_n^1 = 48\frac{q^{2n}}{(1-q^n)^3} - 24\frac{nq^n}{(1-q^n)^2} + 24\frac{q^n}{(1-q^n)^2} + \frac{4n^2+2}{1-q^n} - \frac{48}{1-q^n}\sum_{m=1}^{\infty}\frac{q^m}{(1-q^m)^2}$$

and

$$a_{-n}^1 = -48\frac{q^n}{(1-q^n)^3} + 24\frac{nq^n}{(1-q^n)^2} + 24\frac{q^n}{(1-q^n)^2} - \frac{(4n^2+2)\,q^n}{1-q^n} + 48\frac{q^n}{1-q^n}\sum_{m=1}^{\infty}\frac{q^m}{(1-q^m)^2}.$$

Hence

$$f^3(z) = a_0^1 + 48\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^3}(z^n-z^{-n}) - 24\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)^2}(z^n-z^{-n}) -$$

$$-24\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z^n-z^{-n}) + \sum_{n=1}^{\infty}\frac{(4n^2+2)\,q^n}{1-q^n}(z^n-z^{-n}) + \sum_{n=1}^{\infty}(4n^2+2)\,z^n -$$

$$-48\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(z^n-z^{-n})\right)\left(\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\right) - 48\sum_{n=1}^{\infty}z^n\left(\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\right).$$

Hence, finally we have

$$f^3(z) = a_0' + \sum_{n=1}^{\infty}(4n^2+2)z^n + 48\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^3}(z^n - z^{-n}) -$$

$$\text{(vi)} \qquad -24\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)^2}(z^n - z^{-n}) - 24\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z^n - z^{-n}) + \sum_{n=1}^{\infty}\frac{(4n^2+2)q^n}{1-q^n}(z^n - z^{-n}) -$$

$$-48\,\frac{z}{1-z}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} - 48\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(z^n - z^{-n})\right)\left(\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\right).$$

To find a_0' we proceed as follows:

The constant term a_0' in $f^3(z)$ is the same as the constant term in the product

$$\left[\left(\frac{1+z}{1-z}\right)^2 - 8\sum_{n=1}^{\infty}\frac{q^n(z^n+z^{-n})}{(1-q^n)^2} + 4\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}(z^n+z^{-n}) - 8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\right]$$

$$\times\left[\frac{1+z}{1-z} + 2\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(z^n - z^{-n})\right]$$

or in the product

$$\left(1 + 4\sum_{n=1}^{\infty}nz^n - 8\sum_{n=1}^{\infty}\frac{q^n(z^n+z^{-n})}{(1-q^n)^2} + 4\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}(z^n+z^{-n}) - 8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\right)$$

$$\times\left(1 + 2\sum_{n=1}^{\infty}z^n + 2\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(z^n - z^{-n})\right).$$

The constant term in this product is

$$1-24\sum_{1}^{\infty}\frac{q^n}{(1-q^n)^2}\quad\left(=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right).$$

Hence

$$a_0^1 = 1-24\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\quad\left(=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right).$$

Substituting this value of a_0^1 in (vi) and noticing that

$$1+\sum_{n=1}^{\infty}(4n^2+2)z^n = \left(\frac{1+z}{1-z}\right)^3,$$

we get (v).

2. The Ramanujan identity (3.1) can be employed to obtain the Lambert Series expansions of several combinations of powers η_r, e_s $(r,s=1,2,3)$ and thus we derive the following expansions

$$\frac{\eta_1 e_1}{\omega_1} = \left(\frac{\pi}{\omega_1}\right)^4\left[\frac{1}{72}+2\sum_{m=1}^{\infty}\frac{(2m-1)^2q^{4m-2}}{(1-q^{4m-2})^2}-\frac{5}{3}\sum_{m=1}^{\infty}\frac{m^3q^{2m}}{1-q^{2m}}+\right.$$
$$\left.+\frac{1}{3}\sum_{m=1}^{\infty}\frac{(-1)^m m^3q^{2m}}{(1-q^{2m})}\right],$$

$$\frac{\eta_1 e_2}{\omega_1} = \left(\frac{\pi}{\omega_1}\right)^4\left[-\frac{1}{144}-\frac{5}{3}\sum_{m=1}^{\infty}\frac{m^3q^{2m}}{1-q^{2m}}+\frac{1}{3}\sum_{m=1}^{\infty}\frac{m^3q^m}{1-q^{2m}}+\right.$$
$$\left.+\sum_{m=1}^{\infty}\frac{m^2q^{2m}}{(1-q^{2m})^2}-\sum_{m=1}^{\infty}\frac{m^2q^m}{(1-q^{2m})^2}+\frac{1}{2}\sum_{m=1}^{\infty}\frac{m^2q^m}{1-q^{2m}}\right],$$

$$\frac{\eta_1 e_1^2}{\omega_1} = \left(\frac{\pi}{\omega_1}\right)^6 \left[\frac{1}{432} + \frac{5}{12}\sum_{m=1}^{\infty}\frac{m^4 q^{2m}}{(1-q^{2m})^2} + \frac{1}{12}\sum_{m=1}^{\infty}\frac{(-1)^m m^4 q^{2m}}{(1-q^{2m})^2} - \right.$$

$$\left. - \tfrac{7}{24}\sum_{m=1}^{\infty}\frac{m^5 q^{2m}}{1-q^{2m}} - \tfrac{1}{72}\sum_{m=1}^{\infty}\frac{(-1)^m m^5 q^{2m}}{1-q^{2m}}\right],$$

$$\frac{\eta_1 e_2^2}{\omega_1} = \left(\frac{\pi}{\omega_1}\right)^6 \left[\frac{1}{1728} - \tfrac{1}{24}\sum_{m=1}^{\infty}\frac{m^4 q^m}{1-q^{2m}} + \tfrac{1}{12}\sum_{m=1}^{\infty}\frac{m^4 q^m}{(1-q^{2m})^2} + \right.$$

$$\left. + \tfrac{5}{12}\sum_{m=1}^{\infty}\frac{m^4 q^{2m}}{(1-q^{2m})^2} - \tfrac{1}{72}\sum_{m=1}^{\infty}\frac{m^5 q^m}{1-q^{2m}} - \tfrac{7}{24}\sum_{m=1}^{\infty}\frac{m^5 q^{2m}}{1-q^{2m}}\right].$$

By changing u to $u+\omega_1$ in (3.1) we get

$$\left(\zeta(u+\omega_1) - \eta_1 - \frac{\eta_1 u}{\omega_1}\right)^2 - \wp(u+\omega_1)$$

$$= \left(\frac{2\pi}{\omega_1}\right)^2 \left[-\tfrac{1}{24} + \sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}}{(1-q^{2n})^2}\cos\frac{n\pi u}{\omega_1}\right]. \quad (3.1.1)$$

Left side of (3.1.1) has the following Taylor series expansion at $u=0$

$$\left[-u\left\{\wp(\omega_1) + \eta_1/\omega_1\right\} - \frac{u^3}{3!}\wp^{(2)}(\omega_1) - \frac{u^5}{5!}\wp^{(4)}(\omega_1) - \cdots\right]^2 -$$

$$- \left[\wp(\omega_1) + \frac{u^2}{2!}\wp^{(2)}(\omega_1) + \frac{u^4}{4!}\wp^{(4)}(\omega_1) + \cdots\right].$$

Hence we have from (3.1.1)

(1)
$$
\begin{aligned}
-\wp(\omega_1) &+ \left\{ \left(\wp(\omega_1) + \eta_1/\omega_1 \right)^2 - \frac{\wp^{(2)}(\omega_1)}{2} \right\} u^2 + \\
&+ 1/3 \left\{ \left(\wp(\omega_1) + \eta_1/\omega_1 \right) \wp^{(2)}(\omega_1) - \frac{\wp^{(4)}(\omega_1)}{8} \right\} u^4 + \\
&+ 1/12 \left\{ \frac{1}{5} \left(\wp(\omega_1) + \eta_1/\omega_1 \right) \wp^{(4)}(\omega_1) + 1/3\, \wp^2_{(2)}(\omega_1) - \frac{\wp^{(6)}(\omega_1)}{60} \right\} u^6 + \\
&+ \text{-----} \\
&= \left(\frac{2\pi}{\omega_1} \right)^2 \left[-\frac{1}{24} + \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{(1-q^{2m})^2} \left\{ 1 - 1/2! \left(\frac{m\pi u}{\omega_1} \right)^2 + \frac{1}{4!} \left(\frac{m\pi u}{\omega_1} \right)^4 - + \cdots \right\} \right].
\end{aligned}
$$

Comparing the coefficient of u^2 on either side of (1) we get

$$
\left(\wp(\omega_1) + \eta_1/\omega_1 \right)^2 - \frac{\wp^{(2)}(\omega_1)}{2} = \left(\frac{2\pi}{\omega_1} \right)^4 \frac{1}{8} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^2 q^{2m}}{(1-q^{2m})^2},
$$

or we have

$$
\left(e_1 + \eta_1/\omega_1 \right)^2 - \frac{1}{2} \left(6 e_1^2 - g_2/2 \right) = 2 \left(\frac{\pi}{\omega_1} \right)^4 \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^2 q^{2m}}{(1-q^{2m})^2}.
$$

Hence

(11)
$$
\frac{\eta_1 e_1}{\omega_1} = e_1^2 - g_2/8 - \frac{\eta_1^2}{2\omega_1} + \left(\frac{\pi}{\omega_1} \right)^4 \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^2 q^{2m}}{(1-q^{2m})^2}.
$$

Making use of the Lambert Series expansions of e_1^2, g_2 and that η_1^2, viz.

$$
\frac{\eta_1^2}{\omega_1^2} = \left(\frac{\pi}{\omega_1} \right)^4 \left[\frac{1}{144} + 5/3 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1-q^{2m}} - 2 \sum_{m=1}^{\infty} \frac{m^2 q^{2m}}{(1-q^{2m})^2} \right],
$$

in (ii) we get the simple Lambert Series expansion of $\eta_1 e_1$ noted above.

Comparing next the coefficient of u^4 on either side of (i) we get

$$\left(\wp(\omega_1) + \eta_1/\omega_1\right) \wp^{(2)}(\omega_1) - \frac{\wp^{(4)}(\omega_1)}{8} = \tfrac{1}{2}\left(\frac{\pi}{\omega_1}\right)^6 \sum_{m=1}^{\infty} \frac{(-1)^m m^4 q^{2m}}{(1-q^{2m})^2}.$$

On using

$$\wp^{(4)}(\omega_1) = 48 e_1^3 + 6 g_3, \quad \wp^{(2)}(\omega_1) = 6e_1^2 - g_2/2, \quad g_2 e_1 = 4e_1^3 - g_3,$$

the last relation simplifies to

$$\text{(iii)} \quad \frac{\eta_1 e_1^2}{\omega_1} = \tfrac{1}{3} e_1^3 + \frac{g_2 \eta_1}{12\omega_1} + \tfrac{1}{24} g_3 + \frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^6 \sum_{m=1}^{\infty} \frac{(-1)^m m^4 q^{2m}}{(1-q^{2m})^2}.$$

Now making use of the Lambert Series expansions of e_1^3, g_3 and the Lambert Series expansion for PQ given by Ramanujan [7, p.142], which is equivalent to:

$$\frac{g_2\eta_1}{\omega_1} = \left(\frac{\pi}{\omega_1}\right)^6 \left[\frac{1}{144} + 5\sum_{m=1}^{\infty} \frac{m^4 q^{2m}}{(1-q^{2m})^2} - \tfrac{7}{2}\sum_{m=1}^{\infty} \frac{m^5 q^{2m}}{1-q^{2m}}\right],$$

in (iii) we arrive at the Lambert Series expansion of $\eta_1 e_1^2$ noted above.

Further changing u to ($u+\omega_2$) in the Ramanujan identity (3.1) we get

$$\left(\zeta(u+\omega_2)-\eta_2-\frac{\eta_1 u}{\omega_1}-\frac{\pi i}{\omega_1}\right)^2-\wp(u+\omega_2)$$

$$=\left(\frac{2\pi}{\omega_1}\right)^2\left[-\frac{1}{24}+\frac{1}{2}\sum_{m=1}^{\infty}\frac{q^{3m}}{(1-q^{2m})^2}e^{m\pi u i/\omega_1}+\tfrac{1}{2}\sum_{m=1}^{\infty}\frac{q^m e^{-\frac{m\pi u i}{\omega_1}}}{(1-q^{2m})^2}\right] \tag{3.1.2}$$

Comparing the coefficient of u^2 in the Taylor Expansion of (3.1.2) at $u=0$ we get

$$\left(\wp(\omega_2)+\eta_1/\omega_1\right)^2-\tfrac{1}{2}\wp^{(2)}(\omega_2)$$

$$=-\left(\frac{\pi}{\omega_1}\right)^4\left[\sum_{m=1}^{\infty}\frac{m^2q^{3m}}{(1-q^{2m})^2}-\sum_{m=1}^{\infty}\frac{m^2q^m}{(1-q^{2m})^2}\right].$$

From this relation we get

(iv) $$\frac{\eta_1 e_2}{\omega_1}=e_2^2-g_2/8-\frac{\eta_1^2}{2\omega_1^2}+\left(\frac{\pi}{\omega_1}\right)^4\left[\frac{1}{2}\sum_{m=1}^{\infty}\frac{m^2q^m}{1-q^{2m}}-\sum_{m=1}^{\infty}\frac{m^2q^m}{(1-q^{2m})^2}\right].$$

Making use of the Lambert Series expansions for e_2^2, g_2 and η_1^2 in (iv) we get the expansion for $\eta_1 e_2$ noted above.

Finally, comparing the coefficient of u^4 in (3.1.2) we get

$$\left(\wp(\omega_2)+\eta_1/\omega_1\right)\wp^{(2)}(\omega_2)-\tfrac{1}{8}\wp^{(4)}(\omega_2)$$

$$=\tfrac{1}{4}\left(\frac{\pi}{\omega_1}\right)^6\left[\sum_{m=1}^{\infty}\frac{m^4q^{3m}}{(1-q^{2m})^2}+\sum_{m=1}^{\infty}\frac{m^4q^m}{(1-q^{2m})^2}\right].$$

From this relation we derive the Lambert Series expansion of $\eta_1 e_2^2$ noted above.

The simple Lambert Series expansion for $\eta_1 e_r$ can also be obtained by means of quadratic transformation. In this transformation ω_1 is changed to $\omega_1/2$ keeping ω_2 fixed, so that $q = e^{\pi i \omega_2/\omega_1}$ becomes q^2. By means of this transformation η_1 will be changed to $\eta_1 + \frac{1}{2} e_1 \omega_1$. Denoting this by $\bar{\eta}_1$ we have

$$\bar{\eta}_1 = \eta_1 + \tfrac{1}{2} e_1 \omega_1 .$$

Squaring

$$\bar{\eta}_1^2 = \eta_1 e_1 \omega_1 + e_1^2 \omega_1^2/4 \quad ?$$

Using the series expansions of η_1^2, e_1^2 and $\bar{\eta}_1^2$, we obtain the Lambert Series for $\eta_1 e_1$.

Again, cubing the relation connecting $\bar{\eta}_1$ and η_1 we get

$$\bar{\eta}_1^3 = \eta_1^3 + 3/2\, \eta_1^2 e_1 \omega_1 + 3/4\, \eta_1 e_1^2 \omega_1^2 + 1/8\, e_1^3 \omega_1^3$$

The Lambert series for e_1^3 and η_1^3 (See, Table III in [7, p. 142]) are known. Hence, in case, we know the Lambert series for $\eta_1 e_1^2$ we can derive the Lambert series for $\eta_1^2 e_1$ and vice versa (also see Section 4).

3. An Application

In this connection, we mention an interesting application of the result that we have deducedin the previous section to a classical problem of dynamics treated in Halphen's Fonctions Elliptiques [5]*.

This problem when stated in terms of Elliptic Functions amounts to the following:

If

$$f(e_r) = e_r^2 - \eta_1 e_r/\omega_1 - g_2/6 \qquad (r=1,2,3),$$

then

$$f(e_1) < 0 \quad , \quad f(e_2) > 0 \quad , \quad f(e_3) < 0 \qquad (0<q<1).$$

We give here a simple proof of this using simple series expansions of e_r^2 and $\eta_1 e_r$.

We have

$$f(e_1) = e_1^2 - \eta_1 e_1/\omega_1 - g_2/6 .$$

Hence

$$\left(\frac{\omega_1}{\pi}\right)^4 f(e_1) = 1/36 + 5/3 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1-q^{2m}} + 1/3 \sum_{m=1}^{\infty} \frac{(-1)^m m^3 q^{2m}}{1-q^{2m}} -$$

$$- 1/72 - 2 \sum_{m=1}^{\infty} \frac{(2m-1)^2 q^{4m-2}}{(1-q^{4m-2})^2} + 5/3 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1-q^{2m}}$$

$$- 1/3 \sum_{m=1}^{\infty} \frac{(-1)^m m^3 q^{2m}}{1-q^{2m}} - 1/72 - \frac{10}{3} \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1-q^{2m}} .$$

Hence we have

$$f(e_1) = -2 \left(\frac{\pi}{\omega_1}\right)^4 \sum_{m=1}^{\infty} \frac{(2m-1)^2 q^{4m-2}}{(1-q^{4m-2})^2} .$$

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* For Complete Reference See Introduction

This clearly shows that

$$f(e_1) < 0 \quad , \quad \left(\begin{smallmatrix} 0<q<1 \\ -1<q<0 \end{smallmatrix}\right) .$$

Again

$$f(e_2) \quad = e_2^2 - \eta_1 e_2/\omega_1 - g_2/6 \; ,$$

and hence

$$\left(\frac{\omega_1}{\pi}\right)^4 f(e_2) \;=\; \frac{1}{144} + \frac{5}{3}\sum_{m=1}^{\infty}\frac{m^3 q^{2m}}{1-q^{2m}} + \tfrac{1}{3}\sum_{m=1}^{\infty}\frac{m^3 q^m}{1-q^{2m}} +$$

$$+ \frac{1}{144} + \tfrac{5}{3}\sum_{m=1}^{\infty}\frac{m^3 q^{2m}}{1-q^{2m}} - \tfrac{1}{3}\sum_{m=1}^{\infty}\frac{m^3 q^m}{1-q^{2m}} - \sum_{m=1}^{\infty}\frac{m^2 q^{2m}}{(1-q^{2m})^2} +$$

$$+ \sum_{m=1}^{\infty}\frac{m^2 q^m}{(1-q^{2m})^2} - \tfrac{1}{2}\sum_{m=1}^{\infty}\frac{m^2 q^m}{1-q^{2m}} - \frac{1}{72} - \frac{10}{3}\sum_{m=1}^{\infty}\frac{m^3 q^{2m}}{1-q^{2m}}$$

$$= \; - \sum_{m=1}^{\infty}\frac{m^2 q^{2m}}{(1-q^{2m})^2} + \sum_{m=1}^{\infty}\frac{m^2 q^m}{(1-q^{2m})^2} - \tfrac{1}{2}\sum_{m=1}^{\infty}\frac{m^2 q^m}{1-q^{2m}}$$

$$= \sum_{m=1}^{\infty}\frac{m^2 q^m}{(1+q^m)(1-q^{m})} - \tfrac{1}{2}\sum_{m=1}^{\infty}\frac{m^2 q^m}{1-q^{2m}} = \tfrac{1}{2}\sum_{m=1}^{\infty}\frac{m^2 q^m}{(1+q^m)^2} .$$

Hence we have

$$f(e_2) \;=\; \tfrac{1}{2}\left(\frac{\pi}{\omega_1}\right)^4 \sum_{m=1}^{\infty}\frac{m^2 q^m}{(1+q^m)^2} .$$

This shows

$$f(e_2) > 0 \qquad (0<q<1) .$$

Finally since

$$f(e_1) + f(e_2) + f(e_3) = 0 ,$$

we have

$$f(e_3) = 2\left(\frac{\pi}{\omega_1}\right)^4 \sum_{m=1}^{\infty} \frac{(2m-1)^2 q^{4m-2}}{(1-q^{4m-2})^2} - \frac{1}{2}\left(\frac{\pi}{\omega_1}\right)^4 \sum_{m=1}^{\infty} \frac{m^2 q^m}{(1+q^m)^2} .$$

Now

$$\sum_{m=1}^{\infty} \frac{(2m-1)^2 q^{4m-2}}{(1-q^{4m-2})^2} = \sum_{m=1}^{\infty} \frac{m^2 q^{2m}}{(1-q^{2m})^2} - \sum_{m=1}^{\infty} \frac{(2m)^2 q^{4m}}{(1-q^{4m})^2}$$

$$= \sum_{m=1}^{\infty} \frac{m^2 q^{2m}}{(1-q^{2m})^2} - \sum_{m=1}^{\infty} \frac{m^2 q^{2m}}{(1-q^{2m})^2} + \sum_{m=1}^{\infty} \frac{m^2 q^{2m}}{(1+q^{2m})^2}$$

$$= \sum_{m=1}^{\infty} \frac{m^2 q^{2m}}{(1+q^{2m})^2} = \frac{1}{4} \sum_{m=1}^{\infty} \frac{(2m)^2 q^{2m}}{(1+q^{2m})^2} .$$

Hence

$$f(e_3) = -\frac{1}{2}\left(\frac{\pi}{\omega_1}\right)^4 \sum_{m=1}^{\infty} \frac{(2m-1)^2 q^{2m-1}}{(1+q^{2m-1})^2} .$$

This shows that

$$f(e_3) < 0 , \quad (0 < q < 1) .$$

4. We obtain here the simple Lambert Series expansion of $\eta_1^2 e_1$ by making use of the identity that we obtain by changing u to $u+\omega_1$ in the new identity (3.2). But the Taylor expansion of (3.2) (at $u=0$) as such does not appear to result in any interesting relations except those that are obtained from Ramanujan himself in Table III of Paper no. 18 of his collected works [7, p.143].

Changing u to $u+\omega_1$ in (3.2) we have

$$\left(\frac{\omega_1}{2\pi}\right)^3\left(\zeta(u+\omega_1)-\eta_1-\frac{\eta_1 u}{\omega_1}\right)^3$$

$$= \left(-\frac{1}{4}\tan\frac{\pi u}{\omega_1}\right)^3 - \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}}{(1-q^{2n})^3}\sin\left(\frac{n\pi u}{\omega_1}\right) +$$

$$+\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^n n q^{2n}}{(1-q^{2n})^2}\sin\left(\frac{n\pi u}{\omega_1}\right) + \frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}}{(1-q^{2n})^2}\sin\left(\frac{n\pi u}{\omega_1}\right) +$$

$$-\frac{1}{16}\sum_{n=1}^{\infty}\frac{(-1)^n(2n^2+1)q^{2n}}{1-q^{2n}}\sin\left(\frac{n\pi u}{\omega_1}\right) - \frac{3}{8}\tan\frac{\pi u}{\omega_1}\sum_{n=1}^{\infty}\frac{nq^{2n}}{(1-q^{2n})} +$$

$$+\frac{3}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}}{1-q^{2n}}\sin\left(\frac{n\pi u}{\omega_1}\right)\right)\left(\sum_{n=1}^{\infty}\frac{nq^{2n}}{(1-q^{2n})}\right) \tag{3.2.1}$$

Expanding (3.2.1) in powers of u, the left side is

$$\left(\frac{\omega_1}{2\pi}\right)^3\left\{-u(e_1+\eta_1/\omega_1) - u^3/_{3!}\,\wp^{(2)}(\omega_1) - \frac{u^5}{5!}\wp^{(4)}(\omega_1) - \cdots\right\}^3$$
$$= \left(\frac{\omega_1}{2\pi}\right)^3\left\{-u^3(e_1+\eta_1/\omega_1)^3 - u^5(\quad) - \cdots\right\}.$$

Hence, we have on comparing the coefficient of u^3 in (3.2.1)

$$\left(\frac{\omega_1}{\pi}\right)^6(e_1+\eta_1/\omega_1)^3 = \frac{1}{64} + 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^{2n}}{(1-q^{2n})^3} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^{2n}}{(1-q^{2n})^2} -$$

(1) $$-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4q^{2n}}{(1-q^{2n})^2} + \frac{1}{6}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^5q^{2n}}{1-q^{2n}} + \frac{1}{12}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^{2n}}{1-q^{2n}} +$$

$$+\frac{1}{8}\sum_{n=1}^{\infty}\frac{n\,q^{2n}}{1-q^{2n}} - 2\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^{2n}}{1-q^{2n}}\right)\left(\sum_{n=1}^{\infty}\frac{n\,q^{2n}}{1-q^{2n}}\right).$$

Using now the relations

$$6e_1^2 - g_2/2 = \wp^{(2)}(\omega_1) = \frac{1}{8}\left(\frac{\pi}{\omega_1}\right)^4\left(1 - 16\sum_{m=1}^{\infty}(-1)^{m-1}\frac{m^3q^{2m}}{1-q^{2m}}\right)$$

and

$$\frac{12\,\omega_1\eta_1}{\pi^2} = 1 - 24\sum_{n=1}^{\infty}\frac{n\,q^{2n}}{1-q^{2n}},$$

the terms

$$\frac{1}{12}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^{2n}}{1-q^{2n}} + \frac{1}{8}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} - 2\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^{2n}}{1-q^{2n}}\right)\left(\sum_{n=1}^{\infty}\frac{n\,q^{2n}}{1-q^{2n}}\right)$$

on the right side of (i) simplifies to

$$\text{(ii)} \qquad -3\,\frac{\omega_1^5}{\pi^3}\eta_1 e_1^2 + \tfrac{1}{4}\,\frac{\omega_1^5}{\pi^6}\eta_1 g_2 + \frac{1}{192}\,.$$

We also have the Lambert Series expansions

$$\frac{\eta_1 g_2}{\omega_1} = \left(\frac{\pi}{\omega_1}\right)^6\left[\frac{1}{144} + 5\sum_{m=1}^{\infty}\frac{m^4 q^{2m}}{(1-q^{2m})^2} - \tfrac{7}{2}\sum_{m=1}^{\infty}\frac{m^5 q^{2m}}{1-q^{2m}}\right]$$

$$\text{(iii)} \qquad \frac{\eta_1^3}{\omega_1^3} = \left(\frac{\pi}{\omega_1}\right)^6\left[\frac{1}{1728} + \tfrac{5}{4}\sum_{m=1}^{\infty}\frac{m^4 q^{2m}}{(1-q^{2m})^2} - \tfrac{7}{24}\sum_{m=1}^{\infty}\frac{m^5 q^{2m}}{1-q^{2m}} - 2\sum_{m=1}^{\infty}\frac{m^3 q^{2m}}{(1-q^{2m})^3} + \sum_{m=1}^{\infty}\frac{m^3 q^{2m}}{(1-q^{2m})^2}\right],$$

which are respectively equivalent to the expansions for PQ and P^3 given by Ramanujan [7,p.142]. Substituting (ii), (iii) and the Lambert Series expansion of $\eta_1 e_1^2$ that we have obtained in the earlier section, in (i), we obtain the simple Lambert Series expansion of $\eta_1^2 e_1$, viz.

$$\frac{\eta_1^2 e_1}{\omega_1^2} = \left(\frac{\pi}{\omega_1}\right)^6\left[\frac{1}{864} + \frac{4}{3}\sum_{n=1}^{\infty}\frac{(2n-1)^3 q^{4n-2}}{(1-q^{4n-2})^3} - \tfrac{2}{3}\sum_{n=1}^{\infty}\frac{(2n-1)^3 q^{4n-2}}{(1-q^{4n-2})^2} - \tfrac{1}{6}\sum_{n=1}^{\infty}\frac{(-1)^{n-1} n^4 q^{2n}}{(1-q^{2n})^2} - \tfrac{5}{6}\sum_{n=1}^{\infty}\frac{n^4 q^{2n}}{(1-q^{2n})^2} + \tfrac{1}{72}\sum_{n=1}^{\infty}\frac{(-1)^{n-1} n^5 q^{2n}}{1-q^{2n}} + \tfrac{7}{24}\sum_{1}^{\infty}\frac{n^5 q^{2n}}{1-q^{2n}}\right].$$

Similarly, changing u to $u+\omega_2$ in (3.2) and comparing the coefficient of u^3 in the Taylor expansion of the identity at $u = 0$, one can get the simple Lambert Series expansion of $\eta_1^2 e_2$. The actual expansion is found in the Introduction of the thesis.

5. We derive from (3.2) a new expression for $r_6(n)$ (the number of ways of expressing n as a sum of 6 squares) which in turn is made use of to obtain an interesting arithmetical relation.

The identity (3.2) is also equivalent to

$$\left(\frac{1}{4}\cot\theta/2 + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n\theta\right) = \left(\frac{\cot\theta/2}{4}\right)^3 - \tfrac{3}{2}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^3}\sin n\theta +$$

$$+ \tfrac{3}{4}\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)^2}\sin n\theta + \tfrac{3}{4}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\sin n\theta +$$

$$-\tfrac{1}{16}\sum_{n=1}^{\infty}\frac{(2n^2+1)q^n}{(1-q^n)}\sin n\theta + \tfrac{3}{8}\cot\theta/2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} +$$

$$+ \tfrac{3}{2}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n\theta\right)\left(\sum_{1}^{\infty}\frac{nq^n}{1-q^n}\right). \qquad (3.2.2)$$

Putting $\theta = \pi/2$ in (3.2.2) we get

$$\left(1+2q+2q^4+\cdots\right)^6 = 1 - 96\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{2n-1}}{(1-q^{2n-1})^3} + 48\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{2n-1}}{(1-q^{2n-1})^2} +$$

$$+48\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)q^{2n-1}}{(1-q^{2n-1})^2} - 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)^2q^{2n-1}}{1-q^{2n-1}} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{2n-1}}{1-q^{2n-1}} +$$

$$+ 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 96\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{2n-1}}{1-q^{2n-1}}\right)\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right).$$

(3.2.3)

(3.2.3) gives another expression for $r_6(n)$ which involves a product term.

From the identity (2.3) (Connected with $r_6(n)$) of Part II, the left side of (3.2.3) is also equal to

$$1+16\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}}-4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-1)^2q^{2n-1}}{1-q^{2n-1}}.$$

Now making use of the relations

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^{2n-1}}{(1-q^{2n-1})^3}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{nq^n}{1+q^{2n}}$$

and

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}=\sum_{n=1}^{\infty}\frac{nq^n}{1+q^{2n}},$$

in (3.2.3) we have

(1)
$$\left(24\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^{2n-1}}{1-q^{2n-1}}\right)\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^{2n-1}}{1-q^{2n-1}}+6\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$
$$=16\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-1)^2q^{2n-1}}{1-q^{2n-1}}-12\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-1)q^{2n-1}}{(1-q^{2n-1})^2}.$$

But we have

$$1+4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^{2n-1}}{1-q^{2n-1}}=\left(\frac{2\omega_1}{\pi}\right)(e_1-e_2)^{1/2}=\sum_{n=0}^{\infty}r_2(n)q^n$$

and

$$1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1-q^n} = \frac{12\omega_1}{\pi^2} \eta_1(q) ,$$

where

$$\eta_1(q^2) \equiv \eta_1 .$$

Now using these relations in (i) we have the relation

$$24 \left(\frac{\omega_1}{\pi}\right)^3 \frac{\eta_1(q)}{\omega_1} (e_1 - e_2)^{1/2} = 1 - 64 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^{2n}} - 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2n-1)^2 q^{2n-1}}{1-q^{2n-1}} +$$

$$+ 48 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2n-1) q^{2n-1}}{(1-q^{2n-1})^2} ,$$

which is equivalent to the following arithmetical relation

$$\sigma_1(n)\, r_2(0) + \sigma_1(n-1)\, r_2(1) + \cdots + \sigma_1(1)\, r_2(n-1) - \sigma_1(0)\, r_2(n)$$
$$= 8/3\, a_n + 1/6\, b_n - 2 c_n ,$$

where

$\sigma_1(n)$ = The sum of the divisors of n, $\sigma_1(0) \equiv 1/24$

$r_2(n)$ = The number of ways of expressing n as a sum of two squares

and

$$a_n = \sum_{\substack{dd' = n \\ d' \text{ odd}}} (-1)^{(d'-1)/2} d^2 , \quad b_n = \sum_{\substack{d/n \\ d \text{ odd}}} (-1)^{(d-1)/2} d^2 , \quad c_n = n \sum_{\substack{d/n \\ d \text{ odd}}} (-1)^{(d-1)/2} .$$

6. In this section we obtain (via contour integration) a generalisation of (3.1), viz.

$$\zeta(u)\,\zeta(v) + \zeta(v)\,\zeta(w) + \zeta(w)\,\zeta(u)$$
$$= \tfrac{1}{2}(\eta_1/\omega_1)^2\,(u^2+v^2+w^2) - \eta_1/\omega_1\,\big(u\,\zeta(u) + v\,\zeta(v) + w\,\zeta(w)\big) +$$
$$+\left(\frac{2\pi}{\omega_1}\right)^2\left[1 + 8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\cos\frac{n\pi u}{\omega_1} + 8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\cos\frac{n\pi v}{\omega_1} + \right.$$
$$\left. + 8\sum\frac{q^n}{(1-q^n)^2}\cos\frac{n\pi w}{\omega_1}\right],$$

where

$$u + v + w = 0\,.$$

Let

$$\varphi(z) = f(z)\,f(z\alpha),$$

where

$$f(z) = \frac{1+z}{1-z} + 2\sum_{1}^{\infty}\frac{q^n}{1-q^n}\,(z^n - z^{-n}).$$

$f(z)$ satisfies the functional relation $f(qz) = f(z) + 2$.

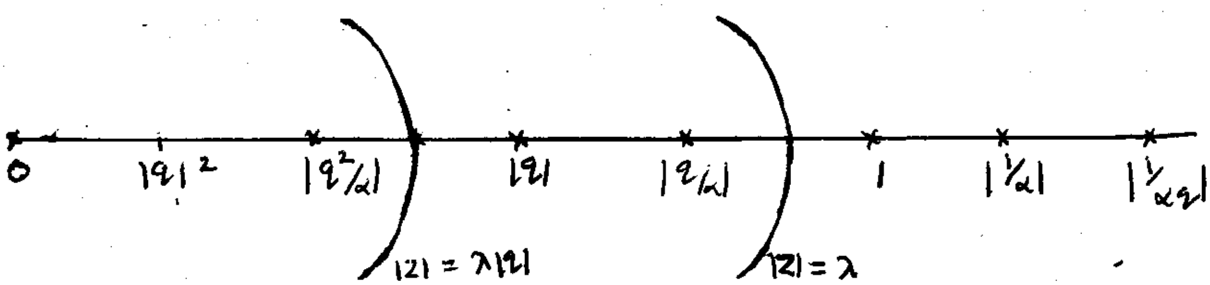
Then

(1) $$\varphi(qz) = \big\{f(z)+2\big\}\big\{f(z\alpha)+2\big\} = \varphi(z) + 2f(z) + 2f(z\alpha) + 4.$$

The function $f(z)$ is analytic in $|q| < |z| < 1$

and $f(z\alpha)$ is analytic in $|q/\alpha| < |z| < |1/\alpha|$.

Let $|q| < |\alpha| < 1$. Then $\varphi(z)$ is analytic in the region $|q/\alpha| < |z| < 1$.



Let
$$\varphi(z) = \sum_{-\infty}^{\infty} C_n z^n \quad , \quad |q/\alpha| < |z| < 1$$
be the Laurant expansion of $\varphi(z)$. Then
$$C_n = \frac{1}{2\pi i}\int_{|z|=\lambda} \frac{\varphi(z)}{z^{n+1}}\, dz \quad .$$
Putting $z = u/q$ in the above integral, we have
$$\frac{C_n}{q^n} = \frac{1}{2\pi i}\int_{|u|=\lambda|q|} \frac{\varphi(u/q)}{u^{n+1}}\, du \quad .$$

Now using the functional relation (1) satisfied by $\varphi(z)$
$$\frac{C_n}{q^n} = \frac{1}{2\pi i}\int_{|u|=\lambda|q|} \frac{\varphi(u) - 2f(u) - 2f(u\alpha) + 4}{u^{n+1}}\, du$$
$$= \frac{1}{2\pi i}\int_{|u|=\lambda|q|} \frac{(f(u)-2)(f(u\alpha)-2)}{u^{n+1}}\, du$$
$$= \frac{1}{2\pi i}\int_{|u|=\lambda} \frac{(f(u)-2)(f(u\alpha)-2)\,du}{u^{n+1}} - R^n,$$
where R^n is the sum of the residues at the poles of $(f(u)-2)(f(u\alpha)-2)/u^{n+1}$ that lie between $|u| = \lambda$ and $|u| = \lambda|q|$.

The function $(f(u)-2)(f(u\alpha)-2)/u^{n+1}$ has only two simple poles at $u = q$ and at $u = q/\alpha$ in this region. Hence

$$R^n = \left(f(q\alpha)-2\right)(-2q)/q^{n+1} + \left(f(q/\alpha)-2\right)(-2q/\alpha)/(q/\alpha)^{n+1}$$

$$= -2f(\alpha)/q^n + 2\alpha^n f(\alpha)/q^n \; .$$

Now we have

$$\frac{c_n}{q^n} = \frac{1}{2\pi i}\int_{|u|=\lambda} \frac{\varphi(u)}{u^{n+1}}\,du - \frac{1}{2\pi i}\int_{|u|=\lambda} \frac{2f(u)}{u^{n+1}}\,du - \frac{1}{2\pi i}\int_{|u|=\lambda} \frac{2f(u\alpha)}{u^{n+1}}\,du - R^n$$

$$= c_n - 4/1-q^n - 4\alpha^n/1-q^n - R^n.$$

Hence

$$c_n = -\frac{4q^n}{(1-q^n)^2} - \frac{4\alpha^n q^n}{(1-q^n)^2} - \frac{R^n q^n}{1-q^n} \; .$$

On using the value of R^n, now we have

$$c_n = -\frac{4q^n}{(1-q^n)^2} - \frac{4\alpha^n q^n}{(1-q^n)^2} + \frac{2f(\alpha)}{1-q^n} - \frac{2\alpha^n f(\alpha)}{1-q^n}, \quad n \neq 0.$$

Changing n to $-n$

$$c_{-n} = -\frac{4q^n}{(1-q^n)^2} - \frac{4\alpha^{-n} q^n}{(1-q^n)^2} - \frac{2q^n f(\alpha)}{1-q^n} + \frac{2\alpha^{-n} f(\alpha)\, q^n}{1-q^n} \quad n \neq 0.$$

The constant term c_0 is obtained by direct multiplication of $f(z)$ and $f(z\alpha)$. Thus we have

$$c_0 = 1 - 4\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}(\alpha^n + \alpha^{-n}) \; .$$

Hence we have

$$\varphi(z) = f(z)f(z\alpha)$$
$$= 1 - 4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(\alpha^n+\alpha^{-n}) - 4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z^n+z^{-n}) -$$
$$-4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z^n\alpha^n+z^{-n}\alpha^{-n}) + 2f(\alpha)\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(z^n-z^{-n}) -$$
$$-2f(\alpha)\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(z^n\alpha^n-z^{-n}\alpha^{-n}) + \frac{2z}{1-z}f(\alpha) - \frac{2\alpha z}{1-\alpha z}f(\alpha).$$

Or we have

$$f(z)f(z\alpha) + f(\alpha)f(z\alpha) - f(z)f(\alpha)$$
$$= 1 - 4\sum_{1}^{\infty}\frac{q^n}{(1-q^n)^2}(z^n+z^{-n}) - 4\sum_{n=1}^{\infty}\frac{q^n(\alpha^n+\alpha^{-n})}{(1-q^n)^2} -$$
$$-4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z^n\alpha^n+z^{-n}\alpha^{-n}).$$

Thus we have proved for $|z_1|<1$ and $|z_2|<1$

$$f(z_1)f(z_2) - f(z_1)f(z_1z_2) - f(z_2)f(z_1z_2)$$
$$= -1 + 4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z_1^n+z_1^{-n}) + 4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z_2^n+z_2^{-n}) \tag{11}$$
$$+4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z_1^nz_2^n+z_1^{-n}z_2^{-n}).$$

Now defining z_3 by the relation $z_1z_2z_3=1$ so that $|q|^2<|z_1z_2| = 1/|z_3| < 1$ then the above relation gives

$$f(z_1)f(z_2)+f(z_2)f(z_3)+f(z_3)f(z_1)$$

$$= -1+4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z_1^n+z_1^{-n})+4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z_2^n+z_2^{-n})+$$

$$+4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(z_3^n+z_3^{-n}).$$

Writing $z_1=e^{i\pi u/\omega_1}$, $z_2=e^{i\pi v/\omega_1}$, $z_3=e^{i\pi w/\omega_1}$ with, $u+v+w=0$, the above relation is equivalent to

$$\zeta(u)\zeta(v)+\zeta(v)\zeta(w)+\zeta(w)\zeta(u)$$

$$= \tfrac{1}{2}\left(\eta_1/\omega_1\right)^2(u^2+v^2+w^2)-\eta_1/\omega_1\left(u\zeta(u)+v\zeta(v)+w\zeta(w)\right)+$$

$$+\left(\frac{2\pi}{\omega_1}\right)^2\Big[-1+8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\cos\frac{n\pi u}{\omega_1}+8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\cos\frac{n\pi v}{\omega_1}+$$

$$+8\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\cos\frac{n\pi w}{\omega_1}\Big],$$

in the ordinary notation of Elliptic functions.

Further, let

$$z_1'=q^r z_1 \quad , z_2'=q^s z_2$$

then using the functional relationship $f(qt)=f(t)+2$ we get from (ii)

$$f(z_1)f(z_2)-f(z_1z_2)f(z_1)-f(z_1z_2)f(z_2)$$

$$-2rf(z_1)-2sf(z_2)-2(r+s)f(z_1z_2)-4(r^2+rs+s^2)$$

$$= -1+4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(q^{rn}z_1^n+q^{-rn}z_1^{-n})+$$

$$+4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(q^{sn}z_2^n+q^{-sn}z_2^{-n})+4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}(q^{(r+s)n}z_1^nz_2^n+q^{-(r+s)n}z_1^{-n}z_2^{-n})$$

7. Algebraic relations connected with $\psi_{r,s}(x)$ and $F_{r,s}(x)$

In this section, we derive some algebraic relations analogous to the relations that are derived from Ramanujan for the function $\Phi_{r,s}(x)$ (symmetric in r and s) in [7, p.140 et seq]. We define two functions $\psi_{r,s}(x)$ and $F_{r,s}(x)$ (not symmetric in r and s) and obtain for them the polynomial expressions involving e_1, e_2 and certain other transcendentals.

Let

$$\psi_{r,s}(x) = \sum_{m} \sum_{n} (-1)^{n-1} m^r n^s x^{mn}, \quad r, s \geq 0$$

so that

$$\psi_{r,s}(x) \neq \psi_{s,r}(x) .$$

We obtain the polynomial expressions for $\psi_{r,s}(x)$ (when $r+s$ is odd) in terms e_1, P_1 and Q_1 where

$$e_1 = \wp(\omega_1)$$

$$P_1 = 1 + 8\,\psi_{0,1}(x) = 1 + 8 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n x^n}{1-x^n}$$

$$Q_1 = 1 - 16\,\psi_{0,3}(x) = 1 - 16 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3 x^n}{1-x^n} .$$

We note that

$$\psi_{0,s}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^s x^n}{1-x^n} \quad ; \quad \psi_{1,s}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^s x^n}{(1-x^n)^2}$$

$$\psi_{r,0}(x) = \sum_{n=1}^{\infty} \frac{m^r x^m}{1+x^m} \quad ; \quad \psi_{r,1}(x) = \sum_{m=1}^{\infty} \frac{m^r x^m}{(1+x^m)^2} .$$

We have from the theory of Elliptic functions

$$\wp(u) = \left(\frac{2\pi}{\omega_1}\right)^2\left[\frac{1}{24} + \left(\frac{1}{4}\operatorname{Cot}\frac{\pi u}{2\omega_1}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}\cos\frac{n\pi u}{\omega_1}\right].$$

Changing u to $u+\omega_1$, in this, we have

$$(1)\quad \wp(u+\omega_1) = \left(\frac{2\pi}{\omega_1}\right)^2\left[\frac{1}{24} + \left(\frac{1}{4}\tan\frac{\pi u}{2\omega_1}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{nx^n}{1-x^n}\cos\frac{n\pi u}{\omega_1}\right].$$

Comparing the like powers of u in the Taylor expansion of (1) at $u=0$, we get successively the following relations:

1. $e_1 = \left(\frac{2\pi}{\omega_1}\right)^2\left[\frac{1}{24} + \psi_{1,0}(x)\right]$
2. $\frac{1}{2}\left(\frac{2\omega_1}{\pi}\right)^4\wp^{(2)}(\omega_1) = 1 - 16\,\psi_{0,3}(x)\ (= Q_1)$
3. $\frac{1}{16}\left(\frac{2\omega_1}{\pi}\right)^6\wp^{(4)}(\omega_1) = 1 + 8\,\psi_{0,5}(x)$
4. $\frac{1}{16}\left(\frac{2\omega_1}{\pi}\right)^8\wp^{(6)}(\omega_1) = 17 - 32\,\psi_{0,7}(x)$
5. $\frac{1}{256}\left(\frac{2\omega_1}{\pi}\right)^{10}\wp^{(8)}(\omega_1) = 31 + 8\,\psi_{0,9}(x)$

etc.

Using now the relations

$$\wp^{(4)}(\omega_1) = 12\,e_1\,\wp^{(2)}(\omega_1) = 24\left(\frac{\pi}{2\omega_1}\right)^4 e_1 Q_1$$

$$\wp^{(6)}(\omega_1) = 36\left\{\wp^{(2)}(\omega_1)\right\}^2 + 12\,\wp(\omega_1)\,\wp^{(4)}(\omega_1)$$

$$= 144\left(\frac{\pi}{2\omega_1}\right)^8 Q_1^2 + 288\left(\frac{\pi}{2\omega_1}\right)^4 e_1^2 Q_1$$

$$\wp^{(8)}(\omega_1) = 180\,\wp^{(2)}(\omega_1)\,\wp^{(4)}(\omega_1) + 12\,\wp(\omega_1)\,\wp^{(6)}(\omega_1)$$
$$= 81\times 2^7\left(\frac{\pi}{2\omega_1}\right)^8 e_1 Q_1^2 + 27\times 2^7\left(\frac{\pi}{2\omega_1}\right)^4 e_1^3 Q_1 ,$$

in the above set of relations, we get the relations in Table I.

Table I

1. $1-16\,\psi_{0,3}(x) = Q_1$
2. $1+8\,\psi_{0,5}(x) = \frac{3}{2}\left(\frac{2\omega_1}{\pi}\right)^2 e_1 Q_1$
3. $17-32\,\psi_{0,7}(x) = 9Q_1^2 + 18\left(\frac{2\omega_1}{\pi}\right)^4 e_1^2 Q_1$
4. $62+16\,\psi_{0,9}(x) = 27\left[3\left(\frac{2\omega_1}{\pi}\right)^2 e_1 Q_1^2 + \left(\frac{2\omega_1}{\pi}\right)^6 e_1^3 Q_1\right]$

etc.

Proceeding in this way, we can express $\psi_{0,2n-1}(x)$, $n = 2,3,\ldots$ as polynomials in e_1 and Q_1.

In order to express $\psi_{1,2n}(x)$ as polynomials in e_1, P_1 and Q_1 we make use of the identity

$$\left[\frac{1}{4}\tan\theta/2 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{1-x^n}\sin n\theta\right]^2 = \left(\frac{1}{4}\tan\theta/2\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{(1-x^n)^2}\cos n\theta + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{nx^n}{1-x^n}\cos n\theta , \tag{11}$$

which is obtained by changing θ to $\pi+\theta$ in the Ramanujan identity [7, pp.138-139] and which is equivalent to (3.1.1).

The Taylor series expansion of (11) at $\theta = 0$ is

$$\left[\frac{1}{8}\cdot P_1\theta + \frac{1}{16} Q_1 \frac{\theta^3}{3!} + \left(\frac{1}{8} + \psi_{0,5}(x)\right)\frac{\theta^5}{5!} + \left(\frac{17}{32} - \psi_{0,7}(x)\right)\frac{\theta^7}{7!} + \right.$$
$$\left. + \cdots \right]^2$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{n x^n}{1-x^n} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(1-x^n)^2} + \frac{1}{2}\psi_{0,1}(x) +$$
$$+ \left(\frac{1}{32} + \psi_{1,2}(x) - \frac{1}{2}\psi_{0,3}(x)\right) \theta^2/2! +$$
$$+ \left(1/16 - \psi_{1,4}(x) + \frac{1}{2}\psi_{0,5}(x)\right) \theta^4/4! +$$
$$+ \left(17/64 + \psi_{1,6}(x) - \frac{1}{2}\psi_{0,7}(x)\right) \theta^6/6! +$$
$$+ \left(31/16 - \psi_{1,8}(x) + \frac{1}{2}\psi_{0,9}(x)\right) \theta^8/8! +$$
$$+ \cdots\cdots$$

Comparing the like powers of θ on either side of this relation, we get successively the relations in Table II.

Table II

1. $32\,\psi_{1,2}(x) = P_1^2 - Q_1$.

2. $32\,\psi_{1,4}(x) = 3\left(\frac{2\omega_1}{\pi}\right)^2 e_1 Q_1 - 2P_1Q_1$.

3. $32\,\psi_{1,6}(x) = 9\left(\frac{2\omega_1}{\pi}\right)^2 e_1 P_1 Q_1 - 9\left(\frac{2\omega_1}{\pi}\right)^4 e_1^2 Q_1 - 2Q_1^2$.

4. $32\,\psi_{1,8}(x) = 39\left(\frac{2\omega_1}{\pi}\right)^2 e_1 Q_1^2 - 36\left(\frac{2\omega_1}{\pi}\right)^4 e_1^2 P_1 Q_1 + 27\left(\frac{2\omega_1}{\pi}\right)^6 e_1^3 Q_1 - 18\,Q_1^2 P_1$.

etc.

Thus, in general, any $\psi_{1,2n}(x)$ can be expressed as a polynomial in e_1, P_1 and Q_1 .

The problem of expressing $\psi_{2n,1}(x)$ as polynomials in e_1, P_1, Q_1 is not as simple as this. We have to consider the expansions of

$$\left(\frac{1}{4}\cot\theta/2+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{1-x^n}\sin n\theta\right)^k, \qquad k\geq 3 \text{ and } k \text{ odd}.$$

Thus, on using

$$\left(\frac{1}{4}\cot\theta/2+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{1-x^n}\sin n\theta\right)^3=\left(\frac{1}{4}\cot\theta/2\right)^3-3/2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n\sin n\theta}{(1-x^n)^3}+$$

(iii)
$$+\,3/4\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{(1-x^n)^2}\sin n\theta+3/4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{nx^n}{(1-x^n)^2}\sin n\theta-$$

$$-\frac{1}{16}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n^2+1)x^n}{1-x^n}\sin n\theta+3/8\cot\theta/2\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}+$$

$$+\frac{3}{2}\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{1-x^n}\sin n\theta\right)\left(\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}\right),$$

(which is obtained by changing θ to $\pi+\theta$ in the relation (3.2.2)) - we get on comparing the coefficient of θ on either side

$$96\,\psi_{2,1}(x)=3\left(\frac{2\omega_1}{\pi}\right)^2 e_1P_1-2Q_1.$$

The other relations that we obtain using (iii) are the polynomial expressions in e_1, P_1 and Q_1 for $\psi_{2,2n-1}(x)$, $n\geq 2$. Thus, for example, the coefficient of θ^3 and θ^5 in the Taylor expansion of (iii) respectively gives

$$64\,\psi_{2,3}(x)=P_1^3-3P_1Q_1+3\left(\frac{2\omega_1}{\pi}\right)^2 e_1Q_1$$

$$64\,\psi_{2,5}(x)=15\left(\frac{2\omega_1}{\pi}\right)^2 e_1P_1Q_1-5P_1^2Q_1-Q_1^2-9\left(\frac{2\omega_1}{\pi}\right)^4 e_1^2Q_1$$

Hence, it appears likely that one gets the polynomial expressions for $\psi_{4,2n-1}(x)$ $(n\geq 1)$ by considering the expansion of

$$\left(\frac{1}{4}\cot\theta/2+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{1-x^n}\sin n\theta\right)^5,$$

and so on.

The polynomial expressions for $\psi_{2r,2s+1}(x)$ ($r\geq 1$, $s\geq 1$, $r\leq s$) can also be derived by considering the differentials

$$x\frac{dP_1}{dx} = 8\,\psi_{1,2}(x) = (P_1^2-Q_1)/4$$

$$x\frac{dQ_1}{dx} = -16\,\psi_{1,4}(x) = P_1Q_1 - 3/2\left(\frac{2\omega_1}{\pi}\right)^2 e_1Q_1$$

$$x\frac{de_1}{dx} = \left(\frac{2\pi}{\omega_1}\right)^2\psi_{2,1}(x) = 1/2\,P_1e_1-\left(\pi/2\omega_1\right)^2 Q_1/3$$

and

$$\psi_{r,s}(x) = \begin{cases}\left(x\dfrac{d}{dx}\right)^r\psi_{0,s-r}(x), & (s+r)\text{ odd},\ s>r\\ \left(x\dfrac{d}{dx}\right)^s\psi_{r-s,0}(x), & (r+s)\text{ odd},\ r>s.\end{cases}$$

Suppose that one can express $\psi_{0,2m+1}(x)$ and $\psi_{2m-1,0}(x)$ ($m = 2,3,\ldots.$) as polynomials in e_1, P_1 and Q_1 then any other $\psi_{r,s}(x)$ $(r+s\text{ odd})$ can be expressed as a polynomial in e_1, P_1 and Q_1 by making use of the above differentials. Table I gives the polynomial expressions for $\psi_{0,2m-1}(x)$. But for the polynomial expressions for $\psi_{2m-1,0}(x)$, We may have to consider successively the trigonometrical series expansions of

$$\left(\frac{1}{4} \tan \theta/2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{1-x^n} \sin n\theta \right)^{2k}, \qquad k \geq 2.$$

Further, the relation connecting $\Phi_{r,s}(x)$ of Ramanujan and $\psi_{r,s}(x)$ is

$$\Phi_{r,s}(x) - \psi_{r,s}(x) = 2^{s+1} \Phi_{r,s}(x^2), \qquad (3.3)$$

where

$$\Phi_{r,s}(x) = \sum_m \sum_n m^r n^s x^{mn}.$$

Hence we have the interesting relation

$$\psi_{r,s}(x) - \psi_{s,r}(x) = \left(2^{r+1} - 2^{s+1}\right) \Phi_{r,s}(x^2).$$

Following Ramanujan $\Phi_{r,s}(x)$ ($r+s$ odd) are expressible as polynomials in P, Q, R where

$$P = 1 - 24 \sum_1^{\infty} \frac{n x^n}{1-x^n}$$

$$Q = 1 + 240 \sum_1^{\infty} \frac{n^3 x^n}{1-x^n}$$

$$R = 1 - 504 \sum_1^{\infty} \frac{n^5 x^n}{1-x^n}.$$

But R being a function of e_1 and Q, $\Phi_{r,s}(x)$ are expressible as polynomials in P, Q and e_1. Since $\Phi_{r,s}(x)$ are polynomials in P, Q, e_1, by means of quadratic transformation $\Phi_{r,s}(x^2)$ are also expressible as polynomials in P, Q, e_1. Hence, in view of the relation (3.3) the functions $\psi_{r,s}(x)$ are polynomials in P, Q and e_1.

Another function similar to the Ramanujan function $\Phi_{r,s}(x)$, but unsymmetric in r and s is

$$F_{r,s}(x) = \sum_{m} \sum_{n} (2m-1)^r n^s x^{(2m-1)n/2} \qquad r,s \geq 0.$$

We express $F_{r,s}(x)$ (when $r+s$ is odd) as polynomials in e_2, P_2 and Q_2, where

$$P_2 = -8\,F_{0,1}(x) = -8 \sum_{m=1}^{\infty} \frac{m\,x^{m/2}}{1-x^m}$$

$$Q_2 = 16\,F_{0,3}(x) = 16 \sum_{m=1}^{\infty} \frac{m^3\,x^{m/2}}{1-x^m}$$

and

$$e_2 = \wp(\omega_2).$$

We note that

$$F_{0,s}(x) = \sum_{n=1}^{\infty} \frac{n^s x^{n/2}}{1-x^n}, \qquad F_{1,s}(x) = 2\sum_{n=1}^{\infty} \frac{n^s x^{n/2}}{(1-x^n)^2} - \sum_{n=1}^{\infty} \frac{n^s x^{n/2}}{1-x^n}$$

$$F_{r,0}(x) = \sum_{m=1}^{\infty} \frac{(2m-1)^r x^{(2m-1)/2}}{1-x^{(2m-1)/2}}, \qquad F_{r,1}(x) = \sum_{m=1}^{\infty} \frac{(2m-1)^r x^{(2m-1)/2}}{\left(1-x^{(2m-1)/2}\right)^2}.$$

From the Theory of Elliptic Functions, we have

(iv) $$\wp(u+\omega_2) = \left(\frac{1}{2\omega_1}\right)^2 \left[-4\eta_1\omega_1 - 8\pi^2 \sum_{m=1}^{\infty} \frac{m\,x^{m/2}}{1-x^m} \cos\frac{m\pi u}{\omega_1}\right].$$

Equating the like powers of u in the Taylor Expansion of (iv) we get successively the relations in Table III.

<u>Table III</u>

1. $e_2 = -\left(\pi/2\omega_1\right)^2\left[1/3 + 8\, F_{1,0}(x)\right]$

2. $16\, F_{0,3}(x) = Q_2$

3. $-16\, F_{0,5}(x) = 3\left(\frac{2\omega_1}{\pi}\right)^2 Q_2 e_2$

4. $32\, F_{0,7}(x) = 9\, Q_2^2 + 18\left(\frac{2\omega_1}{\pi}\right)^4 e_2^2 Q_2$

5. $-16\, F_{0,9}(x) = 81\left(\frac{2\omega_1}{\pi}\right)^2 Q_2^2 e_2 + 27\left(\frac{2\omega_1}{\pi}\right)^6 Q_2 e_2^3$

6. $32\, F_{0,11}(x) = 378\, Q_2^3 + 81\left(\frac{2\omega_1}{\pi}\right)^8 Q_2 e_2^4 + 2187\left(\frac{2\omega_1}{\pi}\right)^4 Q_2^2 e_2^2$

etc.

Another interesting identity connected with the Ramanujan identity (3.1) and is equivalent to (3.1.2) (see page 96) is

$$\text{(v)}\qquad \left(\sum_{n=1}^{\infty}\frac{x^{n/2}}{1-x^n}\sin n\theta\right)^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{n x^n}{1-x^n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(n+1)\,x^{n/2}}{1-x^n}\cos n\theta + \sum_{n=1}^{\infty}\frac{x^{n/2}}{(1-x^n)^2}\cos n\theta,$$

Equating the like powers of θ on either side in the Taylor expansion at $\theta = 0$ of (v) we get successively the relations in Table IV.

<u>Table IV</u>

1. $16\, F_{1,2}(x) = Q_2 - P_2^2$

2. $8\, F_{1,4}(x) = P_2 Q_2 - 3/2\left(\frac{2\omega_1}{\pi}\right)^2 Q_2 e_2$

3. $16\, F_{1,6}(x) = Q_2^2 + 9\left(\frac{2\omega_1}{\pi}\right)^4 Q_2 e_2^2 - 9\left(\frac{2\omega_1}{\pi}\right)^2 Q_2 P_2 e_2$

etc.

In order to express $F_{2n,1}(x)$ as polynomials in e_2, P_2 and Q_2. (v) alone is not sufficient, in addition we have to consider the expansions of

$$\left(\sum_{n=1}^{\infty} \frac{x^{n/2}}{1-x^n} \sin n\theta \right)^k , \qquad k \geq 3 .$$

Thus, on using

$$\text{(vi)} \quad \begin{aligned} \left(\sum_{n=1}^{\infty} \frac{x^{n/2}}{1-x^n} \sin n\theta \right)^3 &= -\tfrac{3}{2} \sum_{n=1}^{\infty} \frac{x^{n/2}}{(1-x^n)^3} \sin n\theta + \tfrac{3}{2} \sum_{n=1}^{\infty} \frac{x^{n/2}}{(1-x^n)^2} \sin n\theta + \\ &+ \tfrac{3}{4} \sum_{n=1}^{\infty} \frac{n x^{n/2}}{(1-x^n)^2} \sin n\theta - \tfrac{1}{8} \sum_{n=1}^{\infty} \frac{(n^2+3n+2) x^{n/2}}{1-x^n} \sin n\theta \\ &+ \tfrac{3}{2} \left(\sum_{n=1}^{\infty} \frac{x^{n/2}}{1-x^n} \sin n\theta \right) \left(\sum_{n=1}^{\infty} \frac{n x^n}{1-x^n} \right), \end{aligned}$$

(which is obtained by changing θ to $\theta + \frac{\omega_2}{2\omega_1}\pi$ and q to q^2 in the relation (3.2.2)) - we get on comparing the coefficient of θ in the Taylor expansion at $\theta=0$ of (vi):

$$12 \, F_{2,1}(x) = Q_2 - 3/2 \left(\frac{2\omega_1}{\pi} \right)^2 e_2 P_2 .$$

The other relations we get on using (vi) are the polynomial expressions in e_2, P_2 and Q_2 for $F_{2,2n-1}(x)$ $(n \geq 2)$ as in the case of $\psi_{r,s}(x)$. For example, the coefficient of θ^3 and θ^5 in the Taylor Expansion of (vi) give respectively

$$16 \, F_{2,3}(x) = 3 P_2 Q_2 - 2 P_2^3 + 3 \left(\frac{2\omega_1}{\pi} \right)^2 e_2 Q_2 .$$

$$\begin{aligned} 16 \, F_{2,5}(x) &= 5 P_2^2 Q_2 + Q_2^2 - 15 \left(\frac{2\omega_1}{\pi} \right)^2 e_2 P_2 Q_2 + \\ &+ 9 \left(\frac{2\omega_1}{\pi} \right)^4 e_2^2 Q_2 . \end{aligned}$$

As in the case of $\psi_{r,s}(x)$ in this case also to obtain the polynomial expressions in e_2, P_2, Q_2 for $F_{2r,2s-1}(x)$ $(r \geq 2, s \geq 1)$ one have to consider the expansions of

$$\left(\sum_{n=1}^{\infty} \frac{x^{n/2}}{1-x^n} \sin n\theta \right)^{2r+1} .$$

These polynomial expressions for $F_{2r,2s+1}(x)$ $(r \geq 1, s \geq 1, r \leq s)$ can also be derived by considering the following differentials

$$x \frac{dQ_2}{dx} = 8 F_{1,4}(x) = P_2 Q_2 - 3/2\left(\frac{2\omega_1}{\pi}\right)^2 e_2 Q_2$$

$$x \frac{dP_2}{dx} = -4 F_{1,2}(x) = (P_2^2 - Q_2)/4$$

$$x \frac{de_2}{dx} = -\left(\frac{\pi}{\omega_1}\right)^2 F_{2,1}(x) = 1/2\, P_2 e_2 - 1/3\left(\frac{\pi}{2\omega_1}\right)^2 Q_2$$

and

$$F_{r,s}(x) = \begin{cases} 2^r \left(x \frac{d}{dx}\right)^r F_{0,s-r}(x) , & (r+s) \text{ odd}, s > r \\ 2^s \left(x \frac{d}{dx}\right)^s F_{r-s,0}(x) , & (s+r) \text{ odd } r > s . \end{cases}$$

Suppose that we can express $F_{2n-1,0}(x)$ and $F_{0,2n+1}(x)$ $(n \geq 1)$ as polynomials in e_2, P_2 and Q_2 then with the help of the above set of differentials we can express the other $F_{r,s}(x)$ $(\overline{r+s} \text{ odd})$ as polynomials in e_2, P_2 and Q_2. Table III gives the polynomial expressions for $F_{0,2n+1}(x)$,

$(n \geq 1)$. But for expressing $F_{2n-1,0}(x)$ as polynomials in e_2, P_2 and Q_2 we may have to consider successively the trigonometric series expansions of

$$\left(\sum_{n=1}^{\infty} \frac{x^{n/2}}{1-x^n} \sin n\theta \right)^{2k}, \qquad k \geq 1$$

Further, we have the following relation connecting $F_{r,s}(x)$ and $\Phi_{r,s}(x)$:

$$F_{r,s}(x^2) = \Phi_{r,s}(x) - 2^r \Phi_{r,s}(x^2)$$

or

$$F_{r,s}(x) = \Phi_{r,s}(x^{1/2}) - 2^r \Phi_{r,s}(x) . \qquad (3.4)$$

Hence, we have the interesting relation

$$F_{r,s}(x) - F_{s,r}(x) = (2^s - 2^r) \Phi_{r,s}(x) .$$

Since $\Phi_{r,s}(x)$ and $\Phi_{r,s}(x^{1/2})$ are polynomials in P, Q and e_2, in view of the relation (3.4) $F_{r,s}(x)$ are also polynomials in P, Q and e_2 .