

A P P E N D I C E S .

APPENDIX A

In proving the formula connected with $\gamma_{2\delta}^{(n)}$
($\delta = 4, 6, 8, 10, 12$) we have noticed that

$$f^{(2n)}(\omega_1) + f^{(2n)}(\omega_2) = (e_1 - e_2)^2 \left[\text{A symmetric function in } e_1 \text{ and } e_2 \right]$$

and

$$f^{(2n)}(\omega_1) - f^{(2n)}(\omega_2) = (e_1 - e_2) \left[\text{A symmetric function in } e_1 \text{ and } e_2 \right]$$

for certain values of n . We prove here that these are true for all $n \geq 1$.

We have

$$f(z + \omega_1) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{f(z) - e_1}, \quad f(z + \omega_2) = e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{f(z) - e_2}$$

and hence

$$(1) \quad f(z + \omega_1) \pm f(z + \omega_2) = (e_1 \pm e_2) + (e_1 - e_2) \left[\frac{e_1 - e_3}{f(z) - e_1} \mp \frac{e_2 - e_3}{f(z) - e_2} \right].$$

Left side and Right side of (1) respectively have the Taylor series expansions

$$(e_1 \pm e_2) + \frac{z^2}{2!} (f^{(2)}(\omega_1) \pm f^{(2)}(\omega_2)) + \frac{z^4}{4!} (f^{(4)}(\omega_1) \pm f^{(4)}(\omega_2)) + \dots$$

and

$$(e_1 \pm e_2) + z^2 (e_1 - e_2) \left[(e_1 - e_3) (1 - e_1 z^2 + c_2 z^4 + c_4 z^6 + \dots)^{-1} \mp \dots \right]$$

at the origin. Hence we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left\{ f^{(2n)}(\omega_1) \pm f^{(2n)}(\omega_2) \right\} \frac{z^{2n-2}}{(2n)!} \\
&= (e_1 - e_2) \left[(e_1 - e_3) \sum_{n=0}^{\infty} (e_1 z^2 - c_2 z^4 - c_4 z^6 - \dots)^n \mp \right. \\
&\quad \left. \mp (e_2 - e_3) \sum_{n=0}^{\infty} (e_2 z^2 - c_2 z^4 - c_4 z^6 - \dots)^n \right] \\
(11) \quad &= (e_1 - e_2) \left[(e_1 - e_3) \left\{ \sum_{n=0}^{\infty} e_1^n z^{2n} - n c_1 e_1^{n-1} z^{2n-2} (c_2 z^4 + c_4 z^6 + \dots) + \right. \right. \\
&\quad \left. \left. + \dots + (-1)^n (c_2 z^4 + c_4 z^6 + \dots)^n \right\} \mp \right. \\
&\quad \left. \mp (e_2 - e_3) \left\{ \sum_{n=0}^{\infty} e_2^n z^{2n} - n c_1 e_2^{n-1} z^{2n-2} (c_2 z^4 + c_4 z^6 + \dots) + \right. \right. \\
&\quad \left. \left. + \dots + (-1)^n (c_2 z^4 + c_4 z^6 + \dots)^n \right\} \right].
\end{aligned}$$

Since $e_1 - e_3 = 2e_1 + e_2$ and $e_2 - e_3 = 2e_2 + e_1$, (11) gives

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left\{ f^{(2n)}(\omega_1) + f^{(2n)}(\omega_2) \right\} \frac{z^{2n-2}}{(2n)!} \\
&= (e_1 - e_2) \sum_{n=0}^{\infty} \left\{ (2e_1^{n+1} - 2e_2^{n+1} + e_1^n e_2 - e_2^n e_1) z^{2n} - \right. \\
&\quad \left. - n c_1 z^{2n-2} (2e_1^n - 2e_2^n + e_1^{n-1} e_2 - e_2^{n-1} e_1) (c_2 z^4 + c_4 z^6 + \dots) + \right. \\
&\quad \left. + \dots + (-1)^{n-1} n c_{n-1} z^2 (2e_1^2 - 2e_2^2) (c_2 z^4 + c_4 z^6 + \dots)^{n-1} + \right. \\
&\quad \left. + (-1)^n (e_1 - e_2) (c_2 z^4 + c_4 z^6 + \dots)^n \right\}.
\end{aligned}$$

This clearly shows that $(e_1 - e_2)^2$ is a factor on the right side. Also, c_2, c_4, \dots being functions of g_2 and g_3 , they are symmetric in e_1 and e_2 .

Comparing the like powers of Z on either side of this relation, we have

$$f^{(2n)}(\omega_1) + f^{(2n)}(\omega_2) = (e_1 - e_2)^2 \times (\text{A symmetric function of } e_1 \text{ and } e_2).$$

We also have from (ii)

$$\begin{aligned} & \sum_{n=1}^{\infty} \left\{ f^{(2n)}(\omega_1) - f^{(2n)}(\omega_2) \right\} \frac{z^{2n-2}}{(2n)!} \\ &= (e_1 - e_2) \sum_{n=0}^{\infty} \left\{ (2e_1^{n+1} + 2e_2^{n+1} + e_1^n e_2 + e_2^n e_1) z^{2n} - \right. \\ & \quad - n c_1 z^{2n-2} (2e_1^n + 2e_2^n + e_1^{n-1} e_2 + e_2^{n-1} e_1) (c_2 z^4 + c_4 z^6 + \dots) + \\ & \quad + \dots + (-1)^{n-1} n c_{n-1} z^2 (2e_1^2 + 2e_2^2 + 2e_1 e_2) (c_2 z^4 + c_4 z^6 + \dots) + \\ & \quad \left. + (-1)^n 3(e_1 + e_2) (c_2 z^4 + c_4 z^6 + \dots)^n \right\}. \end{aligned}$$

This shows that right side has got only $(e_1 - e_2)$ as a factor and the remaining terms are symmetric in e_1 and e_2 .

Comparing the like powers of Z , we have

$$f^{(2n)}(\omega_1) - f^{(2n)}(\omega_2) = (e_1 - e_2) \times (\text{A symmetric function of } e_1 \text{ and } e_2).$$

APPENDIX B

In proving the formulae connected with $\gamma_{2, \delta}^{(n)}$

($\delta = 3, 5, 7, 9, 11$) we have noticed that

$$i \left[\beta^{(2n-1)}\left(\frac{\omega_2}{2}\right) + \beta^{(2n-1)}\left(\frac{\omega_2}{2} + \omega_1\right) \right] - \left[\beta^{(2n-1)}\left(\frac{\omega_1}{2}\right) + \beta^{(2n-1)}\left(\frac{\omega_1}{2} + \omega_2\right) \right]$$

$$(e_1 - e_2)^{3/2} \times (\text{A symmetric function in } e_1 \text{ and } e_2)$$

and

$$i \left[\beta^{(2n-1)}\left(\frac{\omega_2}{2}\right) + \beta^{(2n-1)}\left(\frac{\omega_2}{2} + \omega_1\right) \right] + \left[\beta^{(2n-1)}\left(\frac{\omega_1}{2}\right) + \beta^{(2n-1)}\left(\frac{\omega_1}{2} + \omega_2\right) \right]$$

$$= (e_1 - e_2)^{1/2} \times (\text{A symmetric function in } e_1 \text{ and } e_2)$$

for certain values of n . We prove here that these are true, in general, for all $n \geq 1$.

We have

$$-2 \frac{\beta'(u)}{\sqrt{\beta(u) - e_1}} = \beta(u/2) + \beta(u/2 + \omega_1) - \beta(u/2 + \omega_2) - \beta(u/2 + \omega_3).$$

Putting $u = \omega_2 + t$ in this we have

$$-2 \frac{\beta'(\omega_2 + t)}{\sqrt{\beta(\omega_2 + t) - e_1}} = \beta\left(\frac{\omega_2}{2} + \frac{t}{2}\right) + \beta\left(\frac{\omega_2}{2} + \omega_1 + \frac{t}{2}\right) - \beta\left(\frac{\omega_2}{2} + \omega_2 + \frac{t}{2}\right) - \beta\left(\frac{\omega_2}{2} + \omega_3 + \frac{t}{2}\right).$$

Expanding this in the neighbourhood of the point $t=0$ we get

$$-2 \left[\sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \beta^{(2n)}(\omega_2) \right] \left[(e_2 - e_1) + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \beta^{(2n)}(\omega_2) \right]^{-1/2}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} \left[\beta^{(n)}\left(\frac{\omega_2}{2}\right) + \beta^{(n)}\left(\frac{\omega_2}{2} + \omega_1\right) - \beta^{(n)}\left(\frac{\omega_2}{2} + \omega_2\right) - \beta^{(n)}\left(\frac{\omega_2}{2} + \omega_3\right) \right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{t^{2n-1}}{2^{2n-1} (2n-1)!} \left\{ \beta^{(2n-1)}\left(\frac{\omega_2}{2}\right) + \beta^{(2n-1)}\left(\frac{\omega_2}{2} + \omega_1\right) \right\}.$$

(Here $\beta^0(t) \equiv \beta(t)$)

Hence we have

$$(1) \quad -\frac{1}{(e_2 - e_1)^{1/2}} \left[\sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \rho^{(2n)}(\omega_2) \right] \left[1 + \frac{1}{e_2 - e_1} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \rho^{(2n)}(\omega_2) \right]^{-1/2}$$

$$= \sum_{n=1}^{\infty} \frac{t^{2n-1}}{2^{2n-1} (2n-1)!} \left\{ \rho^{(2n-1)}\left(\frac{\omega_2}{2}\right) + \rho^{(2n-1)}\left(\frac{\omega_2}{2} + \omega_1\right) \right\}.$$

Similarly

$$(11) \quad -\frac{1}{(e_1 - e_2)^{1/2}} \left[\sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \rho^{(2n)}(\omega_1) \right] \left[1 + \frac{1}{e_1 - e_2} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \rho^{(2n)}(\omega_1) \right]^{-1/2}$$

$$= \sum_{n=1}^{\infty} \frac{t^{2n-1}}{2^{2n-1} (2n-1)!} \left\{ \rho^{(2n-1)}\left(\frac{\omega_1}{2}\right) + \rho^{(2n-1)}\left(\frac{\omega_1}{2} + \omega_2\right) \right\}.$$

Since $i(e_2 - e_1)^{1/2} = (e_1 - e_2)^{1/2}$, the relation (1) becomes

$$(111) \quad \frac{1}{(e_1 - e_2)^{1/2}} \left[\sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \rho^{(2n)}(\omega_2) \right] \left[1 + \frac{1}{e_2 - e_1} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \rho^{(2n)}(\omega_2) \right]^{-1/2}$$

$$= i \sum_{n=1}^{\infty} \frac{t^{2n-1}}{2^{2n-1} (2n-1)!} \left[\rho^{(2n-1)}\left(\frac{\omega_2}{2}\right) + \rho^{(2n-1)}\left(\frac{\omega_2}{2} + \omega_1\right) \right].$$

Expanding (ii) we get

$$(iv) \quad \frac{1}{(e_1 - e_2)^{1/2}} \left[\sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \rho^{(2n)}(\omega_1) \right] \left[1 - \frac{1}{2(e_1 - e_2)} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \rho^{(2n)}(\omega_1) + \right.$$

$$\left. + \frac{1 \cdot 3}{2^2 2!} \frac{1}{(e_1 - e_2)^2} \left\{ \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \rho^{(2n)}(\omega_1) \right\}^2 - + \dots + \right.$$

$$\left. + (-1)^r \frac{1 \cdot 3 \cdot \dots \cdot (2r-1)}{2^r \cdot r!} \frac{1}{(e_1 - e_2)^r} \left\{ \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \rho^{(2n)}(\omega_1) \right\}^r + \dots \right]$$

$$= - \sum_{n=1}^{\infty} \frac{t^{2n-1}}{2^{2n-1} (2n-1)!} \left\{ \rho^{(2n-1)}\left(\frac{\omega_1}{2}\right) + \rho^{(2n-1)}\left(\frac{\omega_1}{2} + \omega_2\right) \right\}.$$

Similarly from (iii) we have

$$\begin{aligned}
 & \frac{1}{(e_1 - e_2)^{1/2}} \left[\sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} f^{(2n)}(\omega_2) \right] \left[1 - \frac{1}{2(e_2 - e_1)} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} f^{(2n)}(\omega_2) + \right. \\
 (v) \quad & + \frac{1 \cdot 3}{2^2 \cdot 2!} \frac{1}{(e_2 - e_1)^2} \left\{ \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} f^{(2n)}(\omega_2) \right\}^2 - + \dots + \\
 & + (-1)^r \frac{1 \cdot 3 \cdot \dots \cdot (2r-1)}{2^r \cdot r!} \frac{1}{(e_2 - e_1)^r} \left\{ \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} f^{(2n)}(\omega_2) \right\}^r + \dots \\
 & = i \sum_{n=1}^{\infty} \frac{t^{2n-1}}{2^{2n-1} (2n-1)!} \left\{ f^{(2n-1)}\left(\frac{\omega_2}{2}\right) + f^{(2n-1)}\left(\frac{\omega_2}{2} + \omega_1\right) \right\}.
 \end{aligned}$$

$f^{(2r)}(\omega_1)$ has $(e_1 - e_2)$ as a simple factor for $r \geq 1$. We remove the factor $(e_1 - e_2)$ and compare the like powers of t on either side of (iV). We get a relation of the form

$$(vi) \quad (e_1 - e_2)^{1/2} (k_1 e_1^n + k_2 e_1^{n-1} e_2 + \dots + k_{n+1} e_2^n) = - \left[f^{(2n-1)}\left(\frac{\omega_1}{2}\right) + f^{(2n-1)}\left(\frac{\omega_1}{2} + \omega_2\right) \right],$$

where k_r ($1 \leq r \leq n+1$) are integer constants.

Similarly, from (v) we get

$$(vii) \quad -(e_1 - e_2)^{1/2} (k_1 e_2^n + k_2 e_2^{n-1} e_1 + \dots + k_{n+1} e_1^n) = i \left[f^{(2n-1)}\left(\frac{\omega_2}{2}\right) + f^{(2n-1)}\left(\frac{\omega_2}{2} + \omega_1\right) \right].$$

Adding (vi) and (vii) we have

$$\begin{aligned}
 & i \left[f^{(2n-1)}\left(\frac{\omega_2}{2}\right) + f^{(2n-1)}\left(\frac{\omega_2}{2} + \omega_1\right) \right] - \left[f^{(2n-1)}(\omega_1/2) + f^{(2n-1)}\left(\frac{\omega_1}{2} + \omega_2\right) \right] \\
 & = (e_1 - e_2)^{1/2} \left[l_1 (e_1^n - e_2^n) + l_2 (e_1^{n-2} - e_2^{n-2}) e_1 e_2 + \dots + \right. \\
 & \quad \left. + \begin{cases} l_{n/2} (e_1^2 - e_2^2) e_1^{(n-2)/2} e_2^{(n-2)/2}, & n \text{ even} \\ l_{\frac{n+1}{2}} (e_1 - e_2) e_1^{(n-1)/2} e_2^{(n-1)/2}, & n \text{ odd} \end{cases} \right].
 \end{aligned}$$

where $\gamma_r (1 \leq r \leq n/2 \text{ or } \frac{n+1}{2})$ are integer constants.

This shows that

$$i \left[f^{(2n-1)}\left(\frac{\omega_2}{2}\right) + f^{(2n-1)}\left(\frac{\omega_2 + \omega_1}{2}\right) \right] - \left[f^{(2n-1)}\left(\frac{\omega_1}{2}\right) + f^{(2n-1)}\left(\frac{\omega_1 + \omega_2}{2}\right) \right] \\ = (e_1 - e_2)^{3/2} \times \left[\text{A symmetric function in } e_1 \text{ and } e_2 \right].$$

Subtracting (vi) from (vii) we have

$$i \left[f^{(2n-1)}\left(\frac{\omega_2}{2}\right) + f^{(2n-1)}\left(\frac{\omega_2 + \omega_1}{2}\right) \right] + \left[f^{(2n-1)}\left(\frac{\omega_1}{2}\right) + f^{(2n-1)}\left(\frac{\omega_1 + \omega_2}{2}\right) \right] \\ = (e_1 - e_2)^{1/2} \left[m_1 (e_1^n + e_2^n) + m_2 (e_1^{n-2} + e_2^{n-2}) e_1 e_2 + \dots + \right. \\ \left. + \begin{cases} m_{n/2+1} e_1^{n/2} e_2^{n/2} & , \quad \text{neven} \\ m_{\frac{n+1}{2}} (e_1 + e_2) e_1^{n/2} e_2^{n/2} & , \quad \text{n odd} \end{cases} \right],$$

where $m_r (1 \leq r \leq (n/2 + 1) \text{ or } \frac{n+1}{2})$ are integer constants.

This shows that

$$i \left[f^{(2n-1)}\left(\frac{\omega_2}{2}\right) + f^{(2n-1)}\left(\frac{\omega_2 + \omega_1}{2}\right) \right] + \left[f^{(2n-1)}\left(\frac{\omega_1}{2}\right) + f^{(2n-1)}\left(\frac{\omega_1 + \omega_2}{2}\right) \right] \\ = (e_1 - e_2)^{1/2} \times \left[\text{A symmetric function in } e_1 \text{ and } e_2 \right].$$

Further, ⁱⁿ deducing the formulae connected with $\gamma_{2j}(n)$ ($n = 3, 5, 7, 9, 11$), the successive differentiations of the function $\frac{f'(z)}{\sqrt{p(z) - e_1}}$ can be avoided by making use of (iv) or (v). For purposes of calculation we further simplify (iv) as

$$\begin{aligned}
& \frac{1}{(e_1 - e_2)^{1/2}} \left[t f^{(2)}(\omega_1) + \frac{t^3}{3!} f^{(4)}(\omega_1) + \dots + \frac{t^{2n-1}}{(2n-1)!} f^{(2n)}(\omega_1) + \dots \right] \\
& \times \left[1 - \frac{t^2}{e_1 - e_2} \frac{f^{(2)}(\omega_1)}{2^2} + t^4 \left\{ -\frac{f^{(4)}(\omega_1)}{2 \cdot 4! (e_1 - e_2)} + \frac{1 \cdot 3}{2^5} \frac{f^{(2)}(\omega_1)}{(e_1 - e_2)^2} \right\} + \right. \\
& + t^6 \left\{ -\frac{f^{(6)}(\omega_1)}{2 \cdot 6! (e_1 - e_2)} + \frac{1 \cdot 3}{2^3 \cdot 4!} \frac{f^{(2)}(\omega_1) f^{(4)}(\omega_1)}{(e_1 - e_2)^2} - \frac{1 \cdot 3 \cdot 5}{2^6 \cdot 3!} \frac{f^{(2)}(\omega_1)}{(e_1 - e_2)^3} \right\} + \\
& + t^8 \left\{ -\frac{f^{(8)}(\omega_1)}{2 \cdot 8! (e_1 - e_2)} + \frac{1 \cdot 3}{2^3 \cdot 6!} \frac{f^{(6)}(\omega_1) f^{(2)}(\omega_1)}{(e_1 - e_2)^2} + \frac{1 \cdot 3}{2^3 \cdot (4!)^2} \frac{f^{(4)}(\omega_1)}{(e_1 - e_2)^2} - \right. \\
& \left. - \frac{1 \cdot 3 \cdot 5}{2^6 \cdot 4!} \frac{f^{(4)}(\omega_1) f^{(2)}(\omega_1)}{(e_1 - e_2)^3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^8 \cdot 4!} \frac{f^{(2)}(\omega_1)}{(e_1 - e_2)^4} \right\} + \\
& \left. + \dots \right] \\
& = - \sum_{n=1}^{\infty} \frac{t^{2n-1}}{2^{2n-1} (2n-1)!} \left\{ f^{(2n-1)}\left(\frac{\omega_1}{2}\right) + f^{(2n-1)}\left(\frac{\omega_1}{2} + \omega_2\right) \right\}.
\end{aligned}$$

Comparing the like powers of t in this we get successively the following relations:

$$(1) \quad \frac{f^{(2)}(\omega_1)}{(e_1 - e_2)^{1/2}} = -\frac{1}{2} \left\{ f'(\frac{\omega_1}{2}) + f'(\frac{\omega_1}{2} + \omega_2) \right\}.$$

$$(2) \quad \frac{1}{(e_1 - e_2)^{1/2}} \left\{ f^{(4)}(\omega_1) - 3 \frac{f^{(2)}(\omega_1)}{e_1 - e_2} \right\} = -\frac{1}{2^3} \left\{ f^{(3)}(\frac{\omega_1}{2}) + f^{(3)}(\frac{\omega_1}{2} + \omega_2) \right\}.$$

$$3) \frac{1}{(e_1 - e_2)^{1/2}} \left\{ f^{(6)}(\omega_1) - \frac{15}{2} \frac{f^{(4)}(\omega_1) f^{(2)}(\omega_1)}{e_1 - e_2} + \frac{45}{4} \frac{f^{(2)}(\omega_1)^3}{(e_1 - e_2)^2} \right\} \\ = - \frac{1}{2^5} \left\{ f^{(5)}\left(\frac{\omega_1}{2}\right) + f^{(5)}\left(\frac{\omega_1}{2} + \omega_2\right) \right\}.$$

$$4) \frac{1}{(e_1 - e_2)^{1/2}} \left\{ f^{(8)}(\omega_1) - 14 \frac{f^{(6)}(\omega_1) f^{(2)}(\omega_1)}{(e_1 - e_2)} + \frac{3 \cdot 5}{2} \frac{f^{(4)}(\omega_1)^2}{(e_1 - e_2)} + \right. \\ \left. + \frac{315}{2} \frac{f^{(4)}(\omega_1) f^{(2)}(\omega_1)^2}{(e_1 - e_2)^2} - \frac{1575}{8} \frac{f^{(2)}(\omega_1)^4}{(e_1 - e_2)^3} \right\} \\ = - \frac{1}{2^7} \left\{ f^{(7)}\left(\frac{\omega_1}{2}\right) + f^{(7)}\left(\frac{\omega_1}{2} + \omega_2\right) \right\}.$$

$$5) \frac{1}{(e_1 - e_2)^{1/2}} \left\{ f^{(10)}(\omega_1) - \frac{45}{2} \frac{f^{(8)}(\omega_1) f^{(2)}(\omega_1)}{e_1 - e_2} - 105 \frac{f^{(6)}(\omega_1) f^{(4)}(\omega_1)}{(e_1 - e_2)} + \right. \\ \left. + \frac{945}{2} \frac{f^{(6)}(\omega_1) f^{(2)}(\omega_1)^2}{(e_1 - e_2)^2} + \frac{4725}{4} \frac{f^{(4)}(\omega_1)^2 f^{(2)}(\omega_1)}{(e_1 - e_2)^2} - \right. \\ \left. - \frac{23625}{4} \frac{f^{(4)}(\omega_1) f^{(2)}(\omega_1)^3}{(e_1 - e_2)^3} + \frac{99225}{2^4} \frac{f^{(2)}(\omega_1)^5}{(e_1 - e_2)^4} \right\} \\ = - \frac{1}{2^9} \left\{ f^{(9)}\left(\frac{\omega_1}{2}\right) + f^{(9)}\left(\frac{\omega_1}{2} + \omega_2\right) \right\}.$$

APPENDIX C

In connection with the determination of $\gamma_{2,3}^{(n)}$ ($n=3, 5, 7, 9, 11$) we have made use of the following relation:

$$(1) \quad -2 \frac{\beta^2(2z)}{\sqrt{\beta(2z)-e_1}} = \beta(z) + \beta(z+\omega_1) - \beta(z+\omega_2) - \beta(z+\omega_3).$$

We give here a proof of (1).

Let

$$\varphi_1(z) = \beta(z) + \beta(z+\omega_1) - \beta(z+\omega_2) - \beta(z+\omega_3).$$

Then

$$\begin{aligned} \varphi_1(z+\omega_1) &= \varphi_1(z) \\ \varphi_1(z+\omega_2) &= -\varphi_1(z) \end{aligned}$$

Hence ω_1 and $2\omega_2$ are the periods of $\varphi_1(z)$. The function $\varphi_1(z)$ has poles of second order at $z=0$ and at $z=\omega_2$ in the period parallelogram with vertices $0, \omega_1,$

$2\omega_2+\omega_1, 2\omega_2$.

Let

$$\psi(z) = \frac{\beta^2(z)}{\beta(2z)-e_1}.$$

Then

$$\psi(z) = 4(\beta(2z)-e_2)(\beta(2z)-e_3).$$

Since $\psi(z+\omega_1) = \psi(z)$ and $\psi(z+\omega_2) = \psi(z)$, the function $\psi(z)$ has ω_1 and ω_2 as periods. We consider $\psi(z)$

as having the periods ω_1 and $2\omega_2$ and compare $\varphi_1^2(z)$ with $4\psi(z)$.

The function $\psi(z)$ has poles of 4th order at $z=0$ and at $z=\omega_2$ in the period parallelogram $0, \omega_1, 2\omega_2+\omega_1, 2\omega_2$.

The series expansion of $4\psi(z)$ at $z=0$ is

$$16 \left(\frac{1}{4z^2} - e_2 + \frac{1}{5}g_2z^2 + \dots \right) \left(\frac{1}{4z^2} - e_3 + \frac{1}{5}g_2z^2 + \dots \right).$$

The coefficient of $1/z^2$ in this expansion is

$$-4(e_2 + e_3) = 4e_1.$$

The constant term is

$$8/5 g_2 + 16e_2e_3.$$

Further, the series expansion of $\varphi_1^2(z)$ at $z=0$ is

$$\left\{ \frac{1}{z^2} + (e_1 - e_2 - e_3) + z \frac{1}{2} \left(\beta^{(2)}(\omega_1) - \beta^{(2)}(\omega_2) - \beta^{(2)}(\omega_3) + \frac{1}{10}g_2 \right) + \dots \right\}^2.$$

The coefficient of $1/z^2$ in this expansion is

$$2(e_1 - e_2 - e_3) = 4e_1.$$

The constant term is

$$\begin{aligned} & (e_1 - e_2 - e_3)^2 + \beta^{(2)}(\omega_1) - \beta^{(2)}(\omega_2) - \beta^{(2)}(\omega_3) + \frac{1}{10}g_2 \\ &= 4e_1^2 + 6(e_1^2 - e_2^2 - e_3^2) + 3/5 g_2 \\ &= 4e_1^2 + 12e_2e_3 + 3/5 g_2 \\ &= g_2 + 4e_2e_3 + 12e_2e_3 + 3/5 g_2 = 8/5 g_2 + 16e_2e_3. \end{aligned}$$

Hence the coefficient of $1/z^2$ and the constant term in the series expansion of $\varphi_1^2(z)$ is the same as the coefficient of $1/z^2$ and the constant term in the series

expansion of $4\psi(z)$.

Thus we have

$$4\psi(z) = \varphi_1^2(z) .$$

This gives

$$\varphi_1(z) = \pm \frac{2f'(2z)}{\sqrt{f(2z)-e_1}} .$$

A consideration of the behaviour of the function $f'(u)$ at the origin shows that the lower sign must be taken.

Hence we have (1).

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