CHAPTER

8

CHAPTER 8

*EXPECTED VALUE AND MEAN SQUARE ERROR OF VARIANCE ESTIMATE

In this chapter we shall state a rule of procedure for the estimation of the true error and obtain expressions for bias and mean square error of the estimate thus obtained.

Let us have the following table of the analysis of variance. Then our rule of procedure

Table 8.1

Analysis of Variance : Component of Variance Model

Source of variation	Degree of Freedom	Mean Square	Expected Mean Square
Treatments	n_4		
True Error	n_3	V_3	σ_3^2
Doubtful Error 1	n_2	V_2	σ_2^2
Doubtful Error 2	n_1	V_1	σ_1^2

for the estimation of σ_3^2 is as follows :

* considered in a similar context by Singh (1971).

(i) If $\frac{V_3}{V_2} < \beta_1$ and $\frac{V_{23}}{V_1} < \beta_2$, use $V = V_{123}$;

(ii) If $\frac{V_3}{V_2} < \beta_1$ and $\frac{V_{23}}{V_1} \geq \beta_2$, use $V = V_{23}$;

(iii) If $\frac{V_3}{V_2} \geq \beta_1$, , use $V = V_3$;

where

$\beta_1 = F(n_3, n_2; \alpha_1)$, $\beta_2 = F(n_{23}, n_1; \alpha_2)$, $F(n_i, n_j; \alpha_k)$ refers to the upper 100 α_k % point of the F-distribution with numerator degrees of freedom n_i and denominator degrees of freedom n_j , and V is the estimate proposed for σ_3^2 .

8.1 Expected Value of the Estimate V.

The expected value of V is given by

$$E(V) = E\left(V/\left(\frac{V_3}{V_2} < \beta_1, \frac{V_{23}}{V_1} < \beta_2\right)\right) P\left(\frac{V_3}{V_2} < \beta_1, \frac{V_{23}}{V_1} < \beta_2\right) +$$

$$E\left(V/\left(\frac{V_3}{V_2} < \beta_1, \frac{V_{23}}{V_1} \geq \beta_2\right)\right) P\left(\frac{V_3}{V_2} < \beta_1, \frac{V_{23}}{V_1} \geq \beta_2\right) + \tag{8.1.1}$$

$$E\left(V/\left(\frac{V_3}{V_2} \geq \beta_1\right)\right) P\left(\frac{V_3}{V_2} \geq \beta_1\right) .$$

The joint distribution of V_1, V_2 and V_3 is given by

$$h(V_1, V_2, V_3) = C\, V_1^{n_1/2 - 1}\, V_2^{n_2/2 - 1}\, V_3^{n_3/2 - 1} \tag{8.1.2}$$

$$\exp\left[-\frac{1}{2}\left(\frac{n_1 V_1}{\sigma_1^2} + \frac{n_2 V_2}{\sigma_2^2} + \frac{n_3 V_3}{\sigma_3^2} \right)\right] ,$$

where

$$C = \frac{(n_1/\sigma_1^2)^{n_1/2}\ (n_2/\sigma_2^2)^{n_2/2}\ (n_3/\sigma_3^2)^{n_3/2}}{2^{n_{123}/2}\ \Gamma\frac{n_1}{2}\ \Gamma\frac{n_2}{2}\ \Gamma\frac{n_3}{2}}$$

Now; we evaluate individual components of $E(V)$ in (8.1.1). Let

$$E_1 = E(V/(\frac{V_3}{V_2} < \beta_1,\ \frac{V_{23}}{V_1} < \beta_2))\ P\ (\frac{V_3}{V_2} < \beta_1,\ \frac{V_{23}}{V_1} < \beta_2)$$

$$E_2 = E(V/(\frac{V_3}{V_2} < \beta_1,\ \frac{V_{23}}{V_1} \geq \beta_2))\ P\ (\frac{V_3}{V_2} < \beta_1,\ \frac{V_{23}}{V_1} \geq \beta_2)$$

$$E_3 = E(V/\ \frac{V_3}{V_2} \geq \beta_1)\ P\ (\frac{V_3}{V_2} \geq \beta_1)\ .$$

(a) <u>For E_1</u>

Then to evaluate E_1, we make the following transformation

$$u_1 = n_{123}\ V_{123}$$

$$u_2 = \frac{V_{23}}{V_1}$$

$$u_3 = \frac{V_3}{V_2}$$

in (8.1.1). The Jacobian of transformation is

$$\frac{u_1^2\ u_2\ n_{23}^2}{(n_{23}u_2 + n_1)^3\ (n_3u_3 + n_2)^2}$$

and the joint distribution of u_1, u_2 and u_3 is given by

$$g(u_1, u_2, u_3) = C\,(n_{23})^{n_{23}/2}$$

(8.1.3)
$$\frac{u_1^{n_{123}/2-1}\;u_2^{n_{23}/2-1}\;u_3^{n_3/2-1}}{(n_{23}u_2+n_1)^{n_{123}/2}\,(n_3u_3+n_2)^{n_{23}/2}}$$

$$\exp\left[-\frac{1}{2}\;\frac{n_{23}\,u_1\,u_2}{(n_{23}u_2+n_1)(n_3u_3+n_2)}\left(\frac{n_2}{\sigma_2^2}+\frac{n_3u_3}{\sigma_3^2}+\frac{n_1(n_3u_3+n_2)}{\sigma_1^2\;n_{23}u_2}\right)\right]$$

Then, we have

(8.1.4)
$$E_1 = \frac{1}{n_{123}}\int_{u_3=0}^{\beta_1}\int_{u_2=0}^{\beta_2}\int_{u_1=0}^{\infty} u_1 g(u_1,u_2,u_3)\,du_1\,du_2\,du_3.$$

If we integrate u_1 in (8.1.4) as a gamma variate, we obtain after simplification

$$E_1 = \frac{2^{\frac{n_{123}}{2}+1}\,\Gamma\!\left(\frac{n_{123}}{2}+1\right)C}{n_{123}(n_{23})^{n_1/2+1}}\int_0^{\beta_1}\int_0^{\beta_2}$$

$$\frac{u_3^{\frac{n_3}{2}-1}\,(n_3u_3+n_2)^{\frac{n_1}{2}+1}}{u_2^{n_1/2+2}}\;\frac{(n_{23}u_2+n_1)\,du_2\,du_3}{\left[\frac{n_2}{\sigma_2^2}+\frac{n_3}{\sigma_3^2}u_3+\frac{n_1}{\sigma_1^2}\;\frac{n_3u_3+n_2}{n_{23}\,u_2}\right]^{\frac{n_{123}}{2}+1}}$$

which may be written as

(8.1.5) $$E_1 = K_1 \left[n_{23} I_1 + n_1 I_2 \right],$$

where

(8.1.6) $$K_1 = \frac{(n_1/\sigma_1^2)^{n_1/2} (n_2/\sigma_2^2)^{n_2/2} (n_3/\sigma_3^2)^{n_3/2}}{(n_{23})^{n_1/2+1} B(\frac{n_1}{2}, \frac{n_2}{2}) B(\frac{n_3}{2}, \frac{n_{12}}{2})}$$

(8.1.7) $$I_1 = \int_0^{\beta_1} \int_0^{\beta_2} \frac{u_3^{n_3/2-1}}{u_2^{n_1/2+1}} \frac{(n_3u_3 + n_2)^{n_1/2+1} du_2 du_3}{\left[\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} u_3 + \frac{n_1}{\sigma_1^2} \frac{n_3u_3 + n_2}{n_{23} u_2} \right]^{n_{123}/2+1}}, \text{ and}$$

(8.1.8) $$I_2 = \int_0^{\beta_1} \int_0^{\beta_2} \frac{u_3^{n_3/2-1}}{u_2^{n_1/2+2}} \frac{(n_3u_3 + n_2)^{n_1/2+1} du_2 du_3}{\left[\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} u_3 + \frac{n_1}{\sigma_1^2} \frac{n_3u_3 + n_2}{n_{23} u_2} \right]^{n_{123}/2+1}}$$

We shall first evaluate the integral I_1. Applying the transformation

$$t = \frac{\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} u_3}{\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} u_3 + \frac{n_1}{\sigma_1^2} \frac{n_3 u_3 + n_2}{n_{23}} \frac{1}{u_2}}$$

in (8.1.7), we get

$$I_1 = \int_0^{\beta_1} \int_0^{k_2} \left(\frac{n_{23}\, \sigma_1^2}{n_1} \right)^{n_1/2}$$

$$\frac{u_3^{n_3/2 - 1} (n_3 u_3 + n_2)\; t^{n_{23}/2}}{\left(\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} u_3\right)^{n_{23}/2 + 1}} (1-t)^{n_1/2 - 1} \, dt \, du_3 \, ,$$

where

(8.1.9) $$k_2 = \frac{\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} u_3}{\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} u_3 + \frac{n_1}{\sigma_1^2} \frac{n_3 u_3 + n_2}{n_{23}\, \beta_2}}$$

If we expand $(1-t)^{n_1/2 - 1}$ in the above by the Binomial Theorem, integrate with respect to t and simplify, we get

$$I_1 = \left(\frac{n_{23}\sigma_1^2}{n_1}\right)^{\frac{n_1}{2}} \left[\sum_{i=0}^{n_1/2 - 1} \binom{n_1/2 - 1}{i} \frac{(-1)^i}{\left(\frac{n_{23}}{2} + i + 1\right)} \right.$$

(8.1.10)

$$\left(\frac{n_3}{\sigma_3^2}\right)^i \left[\sum_{j=0}^{i} \binom{i}{j} \left(\frac{n_2\,\sigma_3^2}{n_3\,\sigma_2^2}\right)^j (n_3\, I_{11} + n_2\, I_{12}) \right] \Bigg]$$

(8.1.10)

where

(8.1.11) $$I_{11} = \int_0^{\beta_1} \frac{u_3^{n_3/2+i-j}\, du_3}{\left[\frac{n_2}{\sigma_2^2} k_4 + \frac{n_3}{\sigma_3^2} k_3 u_3\right]^{n_{23}/2+i+1}} ,$$

(8.1.12) $$I_{12} = \int_0^{\beta_1} \frac{u_3^{n_3/2+i-j-1}\, du_3}{\left[\frac{n_2}{\sigma_2^2} k_4 + \frac{n_3}{\sigma_3^2} k_3\, u_3\right]^{n_{23}/2+i+1}} ,$$

(8.1.13) $$k_3 = \left(1 + \frac{n_1}{n_{23}\,\beta_2}\,\varphi_{31}\right), \quad k_4 = \left(1 + \frac{n_1}{n_{23}\,\beta_2}\,\varphi_{21}\right)$$

$$\text{and } \varphi_{ij} = \frac{\sigma_i^2}{\sigma_j^2}$$

To evaluate I_{11}, we make the substitution

$$x = \frac{\frac{n_3}{\sigma_3^2} k_3\, u_3}{\frac{n_2}{\sigma_2^2} k_4 + \frac{n_3}{\sigma_3^2} k_3 u_3}$$

in (8.1.11) and obtain after some simplifications.

$$I_{11} = \frac{1}{\left(\frac{n_3}{\sigma_3^2} k_3\right)^{n_3/2 + i - j + 1} \left(\frac{n_2}{\sigma_2^2} k_4\right)^{n_2/2 + j}}$$

$$\int_0^{X_1} x^{n_3/2 + i - j} (1-x)^{n_2/2 + j - 1} dx$$

which is obviously

$$(8.1.14) \qquad I_{11} = \frac{B_{X_1}\left(\frac{n_3}{2} + i - j + 1, \frac{n_2}{2} + j\right)}{\left(\frac{n_3}{\sigma_3^2} k_3\right)^{n_3/2 + i - j + 1} \left(\frac{n_2}{\sigma_2^2} k_4\right)^{n_2/2 + j}}$$

where

$$(8.1.15) \qquad X_1 = \frac{\frac{n_3}{\sigma_3^2} k_3 \beta_1}{\frac{n_2}{\sigma_2^2} k_4 + \frac{n_3}{\sigma_3^2} k_3 \beta_1}$$

If we follow the same procedure as in the evaluation of I_{11}, we obtain

$$(8.1.16) \qquad I_{12} = \frac{B_{X_1}\left(\frac{n_3}{2} + i - j, \frac{n_2}{2} + j + 1\right)}{\left(\frac{n_3}{\sigma_3^2} k_3\right)^{n_3/2 + i - j} \left(\frac{n_2}{\sigma_2^2} k_4\right)^{n_2/2 + j + 1}}$$

Substituting the values of I_{11} and I_{12} from (8.1.14) and (8.1.16) in (8.1.10), we obtain

$$I_1 = \frac{(n_{23})^{n_1/2}}{\left(\frac{n_1}{\sigma_1^2}\right)^{n_1/2}\left(\frac{n_2}{\sigma_2^2}\right)^{n_2/2}\left(\frac{n_3}{\sigma_3^2}\right)^{n_3/2} k_3^{n_3/2}\, k_4^{n_2/2}}$$

(8.1.17)

$$\left[\sum_{i=0}^{\frac{n_1}{2}-1}\binom{\frac{n_1}{2}-1}{i}\frac{(-1)^i}{\left(\frac{n_{23}}{2}+i+1\right)}\frac{1}{k_3^{\,i}}\left[\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^j\right.\right.$$

$$\left[\frac{\sigma_3^2}{k_3}B_{x_1}\left(\frac{n_3}{2}+i-j+1,\ \frac{n_2}{2}+j\right)+\right.$$

$$\left.\left.\left.+\frac{\sigma_2^2}{k_4}B_{x_1}\left(\frac{n_3}{2}+i-j,\ \frac{n_2}{2}+j+1\right)\right]\right]\right]$$

Integrating on similar lines, we get

$$I_2 = \frac{(n_{23})^{n_1/2+1}}{\left(\frac{n_1}{\sigma_1^2}\right)^{n_1/2+1}\left(\frac{n_2}{\sigma_2^2}\right)^{n_2/2}\left(\frac{n_3}{\sigma_3^2}\right)^{n_3/2} k_3^{n_3/2}\, k_4^{n_2/2}}$$

(8.1.18)

$$\left[\sum_{i=0}^{\frac{n_1}{2}-1}\binom{\frac{n_1}{2}-1}{i}\frac{(-1)^i}{\left(\frac{n_{23}}{2}+i\right)k_3^{i}}\left[\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^j B_{x_1}\left(\frac{n_3}{2}+i-j,\frac{n_2}{2}+j\right)\right]\right]$$

Substituting the values of I_1 and I_2 from (8.1.17) and (8.1.18) in (8.1.5) and simplifying, we obtain

$$E_1 = \frac{1}{k_5 k_3^{\frac{n_3}{2}} k_4^{\frac{n_2}{2}}} \left[\left[\sum_{i=0}^{\frac{n_1}{2}} \binom{\frac{n_1}{2}-1}{i} \frac{(-1)^i}{(\frac{n_{23}}{2}+i+1) k_3^i} \right.\right. \tag{8.1.19}$$

$$\left[\sum_{j=0}^{i} \binom{i}{j} \left(\frac{k_3}{k_4}\right)^j \left(\frac{\sigma_3^2}{k_3} B_{X_1}\left(\frac{n_3}{2}+i-j+1, \frac{n_2}{2}+j\right) + \right.\right.$$

$$\left.\left.\left. + \frac{\sigma_2^2}{k_4} B_{X_1}\left(\frac{n_3}{2}+i-j, \frac{n_2}{2}+j+1\right)\right)\right]\right] + \left[\sum_{i=0} \binom{\frac{n_1}{2}-1}{i} \right.$$

$$\frac{(-1)^i}{(\frac{n_{23}}{2}+i) k_3^i} \left[\sum_{j=0}^{i} \binom{i}{j} \left(\frac{k_3}{k_4}\right)^j \sigma_1^2 B_{X_1}\left(\frac{n_3}{2}+i-j, \right.\right.$$

$$\left.\left.\left. \frac{n_2}{2}+j\right) \right]\right]\right]$$

where

$$k_5 = B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) B\left(\frac{n_3}{2}, \frac{n_{12}}{2}\right). \tag{8.1.20}$$

(b) <u>For E_2</u>

To evaluate E_2, we apply the transformation

$$w_1 = n_{23} V_{23}$$

$$w_2 = \frac{V_{23}}{V_1}$$

$$w_3 = \frac{V_3}{V_2}$$

in the joint density of V_1, V_2, V_3 given by (8.1.2). The joint distribution of w_1, w_2 and w_3 is then given by

$$J(w_1, w_2, w_3) = C \frac{w_1^{n_{123}/2 - 1} \; w_3^{n_3/2 - 1}}{(n_{23})^{n_1/2} (n_3 w_3 + n_2)^{n_{23}/2} \; w_2^{n_1/2 + 1}}$$

$$\exp \left[-\frac{1}{2} \frac{w_1}{(n_3 w_3 + n_2)} \left(\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3 + \frac{n_1}{\sigma_1^2} \; \frac{n_3 w_3 + n_2}{n_{23} w_2} \right) \right]$$

Hence

$$E_2 = \int_0^{\beta_1} \int_{\beta_2}^{\infty} \int_0^{\infty} w_1 \; J(w_1, w_2, w_3) dw_1 \; dw_2 \; dw_3$$

If we integrate w_1 in the above, we obtain after simplification

(8.1.21) $$E_2 = K_6 \int_0^{\beta_1} \int_{\beta_2}^{\infty}$$

$$\frac{w_3^{n_3/2 - 1} (n_3 w_3 + n_2)^{n_1/2 + 1} \; dw_2 \; dw_3}{w_2^{n_1/2 + 1} \left[\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3 + \frac{n_1}{\sigma_1^2} \; \frac{n_3 w_3 + n_2}{n_{23} w_2} \right]^{n_{123}/2 + 1}}$$

where

(8.1.22) $$K_6 = n_{123} K_1$$

Now, we make the substitution

$$t = \frac{\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3}{\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3 + \frac{n_1}{\sigma_1^2} \frac{n_3 w_3 + n_2}{n_{23}} \frac{1}{w_2}}$$

in (8.1.21), and get

$$E_2 = K_6 \int_0^{P_1} \int_{k_2}^{1} \left(\frac{n_{23}\sigma_1^2}{n_1} \right)^{n_1/2}$$

$$\frac{w_3^{n_3/2 - 1} (n_3 w_3 + n_2) t^{n_{23}/2} (1-t)^{n_1/2 - 1} \, dt \, dw_3}{\left(\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3\right)^{n_{23}/2 + 1}}$$

Expanding $(1-t)^{n_1/2 - 1}$ by Binomial Theorem, integrating with respect to t and simplifying, we obtain

$$(8.1.23) \qquad E_2 = \left(\frac{n_{23}}{n_1} \sigma_1^2 \right)^{n_1/2} K_6 \sum_{i=0}^{n_1/2 - 1} \binom{n_1/2 - 1}{i}$$

$$\frac{(-1)^i}{\left(\frac{n_{23}}{2} + i + 1\right)} \left[n_3 I_1 + n_2 I_2 - I_3 \right]$$

where

$$(8.1.24) \qquad I_1 = \int_0^{\beta_1} \frac{w_3^{n_3/2} \, dw_3}{\left(\frac{n_2}{\sigma_2^2} + \frac{n_3 w_3}{\sigma_3^2}\right)^{n_{23}/2 + 1}}$$

$$(8.1.25) \qquad I_2 = \int_0^{\beta_1} \frac{w_3^{n_3/2 - 1} \, dw_3}{\left(\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3\right)^{n_{23}/2 + 1}}$$

$$(8.1.26) \qquad I_3 = \int_0^{\beta_1} \frac{w_3^{n_3/2 - 1} (n_3 w_3 + n_2) \left(\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3\right)^1 dw_3}{\left(\frac{n_2}{\sigma_2^2} k_4 + \frac{n_3}{\sigma_3^2} k_3 w_3\right)^{n_{23}/2 + 1 + 1}}$$

we shall first solve I_1. If we apply the substitution

$$x = \frac{\frac{n_3}{\sigma_3^2} w_3}{\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3}$$

in (8.1.24) and integrate with respect to x, we get after simplification

$$(8.1.27) \qquad I_1 = \frac{B_{x_{21}}\left(\frac{n_3}{2} + 1, \frac{n_2}{2}\right)}{\left(\frac{n_3}{\sigma_3^2}\right)^{n_3/2 + 1} \left(\frac{n_2}{\sigma_2^2}\right)^{n_2/2}} \, ,$$

where

$$X_{21} = \frac{n_3 \sigma_2^2 \beta_1}{n_2 \sigma_3^2 + n_3 \sigma_2^2 \beta_1} \qquad (6.1.28)$$

Similarly, we get for I_2

$$I_2 = \frac{B_{X_{21}}(\frac{n_3}{2}, \frac{n_2}{2} + 1)}{(\frac{n_3}{\sigma_3^2})^{n_3/2} (\frac{n_2}{\sigma_2^2})^{n_2/2 + 1}} \qquad (6.1.29)$$

To solve I_3, we expand $(\frac{n_2}{\sigma_2^2} + \frac{n_3}{\sigma_3^2} w_3)^i$ by Binomial Theorem and obtain

$$I_3 = \sum_{j=0}^{i} \binom{i}{j} (\frac{n_2}{\sigma_2^2})^j \int_0^{\beta_1} (\frac{n_3}{\sigma_3^2})^{i-j}$$

$$\frac{w_3^{n_3/2 + i - j - 1} (n_3 w_3 + n_2)\, dw_3}{(\frac{n_2}{\sigma_2^2} k_4 + \frac{n_3}{\sigma_3^2} k_3 w_3)^{n_{23}/2 + i + 1}}$$

which may be written as

$$I_3 = \sum_{j=0}^{i} \binom{i}{j} \left(\frac{n_2}{n_3}\frac{\sigma_3^2}{\sigma_2^2}\right)^j \left(\frac{n_3}{\sigma_3^2}\right)^i \left[n_3 I_{31} + n_2 I_{32}\right], \tag{8.1.30}$$

where

$$I_{31} = \int_0^{\beta_1} \frac{w_3^{n_3/2 + i - j} \, dw_3}{\left(\frac{n_2}{\sigma_2^2} k_4 + \frac{n_3}{\sigma_3^2} k_3 w_3\right)^{n_{23}/2 + i + 1}} = I_{11}$$

$$I_{32} = \int_0^{\beta_1} \frac{w_3^{n_3/2 + i - j - 1} \, dw_3}{\left(\frac{n_2}{\sigma_2^2} k_4 + \frac{n_3}{\sigma_3^2} k_3 w_3\right)^{n_{23}/2 + i + 1}} = I_{12}$$

Substituting for I_{31} and I_{32} in (8.1.30), we get

$$I_3 = \frac{(\sigma_2^2/n_2)^{n_2/2} (\sigma_3^2/n_3)^{n_3/2}}{k_3^{n_3/2 + i} \, k_4^{n_2/2}} \sum_{j=0}^{i} \binom{i}{j} \left(\frac{k_3}{k_4}\right)^j \tag{8.1.31}$$

$$\left(\frac{\sigma_3^2}{k_3} B_{x_1}\left(\frac{n_3}{2} + i - j + 1, \; \frac{n_2}{2} + j\right)\right.$$

$$+ \frac{\sigma_2^2}{k_4} B_{X_1}\left(\frac{n_3}{2} + i-j, \frac{n_2}{2} + j+1\right)\Bigg)$$

If we now substitute the values of I_1, I_2 and I_3 from (8.1.27), (8.1.29) and (8.1.31) in (8.1.23), we obtain

$$E_2 = \frac{n_{123}}{n_{23}^2\, B(\frac{n_1}{2}, \frac{n_{23}}{2})} \Bigg[\sum_{i=0}^{n_1/2-1} \binom{n_1/2-1}{i} \frac{(-1)^i}{(\frac{n_{23}}{2}+i+1)}$$

$$\Big[n_3\ \sigma_3^2\ I_{X_{21}}\left(\frac{n_3}{2}+1, \frac{n_2}{2}\right) +$$

$$n_2\ \sigma_2^2\ I_{X_{21}}\left(\frac{n_3}{2}, \frac{n_2}{2}+1\right)\Big]\Bigg]$$

$$- \frac{n_{123}}{n_{23}\, k_5\, k_3^{n_3/2}\, k_4^{n_2/2}} \Bigg[\sum_{i=0}^{n_1/2-1} \binom{n_1/2-1}{i}$$

(8.1.32)

$$\frac{(-1)^i}{(\frac{n_{23}}{2}+i+1)k_3^i} \left\{ \sum_{j=0}^{i} \binom{i}{j} \left(\frac{k_3}{k_4}\right)^j \right.$$

$$\left(\frac{\sigma_3^2}{k_3} B_{X_1}\left(\frac{n_3}{2}+i-j+1, \frac{n_2}{2}+j\right) + \frac{\sigma_2^2}{k_4} \right.$$

$$\left. \left. B_{X_1}\left(\frac{n_3}{2}+i-j, \frac{n_2}{2}+j+1\right) \right) \right\} \Bigg]$$

Since $\sum_{i=0}^{n_1/2-1} \binom{n_1/2-1}{i} \frac{(-1)^i}{(\frac{n_{23}}{2}+i+1)} = B(\frac{n_{23}}{2}+1, \frac{n_1}{2})$

(8.1.32) becomes

$$E_2 = \frac{n_3 \sigma_3^2 I_{X_{21}}(\frac{n_3}{2}+1, \frac{n_2}{2}) + n_2 \sigma_2^2 I_{X_{21}}(\frac{n_3}{2}, \frac{n_2}{2}+1)}{n_{23}}$$

$$- \frac{n_{123}}{n_{23} k_5 k_3^{n_3/2} k_4^{n_2/2}} \left[\sum_{i=0}^{n_1/2-1} \binom{n_1/2-1}{i} \right.$$

(8.1.33)

$$\frac{(-1)^i}{(\frac{n_{23}}{2}+i+1)k_3^i} \left[\sum_{j=0}^{i} \binom{i}{j} (\frac{k_3}{k_4})^j \right.$$

$$\left[\frac{\sigma_3^2}{k_3} B_{X_1}(\frac{n_3}{2}+i-j+1, \frac{n_2}{2}+j) + \right.$$

$$\left. \left. \left. + \frac{\sigma_2^2}{k_4} B_{X_1}(\frac{n_3}{2}+i-j, \frac{n_2}{2}+j+1) \right] \right] \right]$$

(c) <u>For E_3</u>

To evaluate E_3 we integrate out V_1 in the joint distribution of V_1, V_2 and V_3 and get the joint density of V_2 and V_3 as follows :

$$f(V_2, V_3) = C_1 V_2^{n_2/2 - 1} V_3^{n_3/2 - 1} \exp\left[-\frac{1}{2}\left(\frac{n_2 V_2}{\sigma_2^2} + \frac{n_3 V_3}{\sigma_3^2}\right)\right] \quad (8.1.34)$$

where

$$C_1 = \frac{(n_2/\sigma_2^2)^{n_2/2} (n_3/\sigma_3^2)^{n_3/2}}{2^{n_{23}/2} \Gamma\frac{n_2}{2} \Gamma\frac{n_3}{2}}$$

Let us apply the transformation

$$y_1 = V_3$$

$$y_2 = \frac{V_3}{V_2}$$

in (8.1.34). Then the joint distribution of y_1 and y_2 is given by

$$h(y_1, y_2) = C_1 \frac{y_1^{n_{23}/2 - 1}}{y_1^{n_2/2 + 1}} \exp\left[-\frac{1}{2} y_1\left(\frac{n_2}{\sigma_2^2 y_2} + \frac{n_3}{\sigma_3^2}\right)\right]$$

Hence

$$E_3 = \int_{F_1}^{\infty} \int_{0}^{\infty} y_1 \, h(y_1, y_2) \, dy_1 \, dy_2$$

If we integrate out y_1 as a gamma variate in the above, we obtain after simplification

(8.1.35) $$E_3 = K_7 \int_{\beta_1}^{\infty} \frac{dy_2}{y_2^{n_2/2+1} \left(\frac{n_2}{\sigma_2^2 y_2} + \frac{n_3}{\sigma_3^2} \right)^{n_{23}/2+1}} ,$$

where

(8.1.36) $$K_7 = \frac{n_{23} \left(\frac{n_2}{\sigma_2^2} \right)^{n_2/2} \left(\frac{n_3}{\sigma_3^2} \right)^{n_3/2}}{B\left(\frac{n_2}{2}, \frac{n_3}{2} \right)}$$

If we make the substitution

$$t = \frac{n_2 \sigma_3^2}{n_2 \sigma_3^2 + n_3 \sigma_2^2 y_2}$$

in (8.1.35), we get

(8.1.37) $$E_3 = \sigma_3^2 \, I_{X_3}\left(\frac{n_2}{2}, \frac{n_3}{2} + 1 \right)$$

where

(8.1.38) $$X_3 = 1 - X_{21}$$

Finally, the expected value of V is obtained by adding (8.1.19), (8.1.33) and (8.1.37) and if expressed as a fraction of σ_3^2, it is given by

(8.1.39)

$$\frac{E(V)}{\sigma_3^2} = 1 + \frac{n_2}{n_{23}}\left[\varphi_{23}\, I_{X_{21}}\left(\frac{n_3}{2}, \frac{n_2}{2}+1\right) - I_{X_{21}}\left(\frac{n_3}{2}+1, \frac{n_2}{2}\right)\right] + \frac{\varphi_{13}}{k_5 k_3^{n_3/2}\, k_4^{n_2/2}}$$

$$\left[\sum_{i=0}^{n_1/2-1}\binom{n_1/2-1}{i}\frac{(-1)^i}{\left(\frac{n_{23}}{2}+i\right)k_3^{\,i}}\left[\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^j\right.\right.$$

$$\left.\left. B_{X_1}\left(\frac{n_3}{2}+i-j, \frac{n_2}{2}+j\right)\right]\right] - \frac{n_1}{n_{23}\, k_5\, k_3^{n_3/2}\, k_4^{n_2/2}}$$

$$\left[\sum_{i=0}^{n_1/2-1}\binom{n_1/2-1}{i}\frac{(-1)^i}{\left(\frac{n_{23}}{2}+i+1\right)}\ \frac{1}{k_3^{\,i}}\left[\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^j\right.\right.$$

$$\left.\left.\left[\frac{B_{X_1}\left(\frac{n_3}{2}+i-j+1, \frac{n_2}{2}+j\right)}{k_3} + \frac{\varphi_{23}}{k_4} B_{X_1}\left(\frac{n_3}{2}+i-j, \frac{n_2}{2}+j+1\right)\right]\right]\right]$$

<u>**Special Cases.**</u>

<u>Case 1</u>. For $\beta_1 = \beta_2 = 0$, that is, when we never pool the mean squares V_1, V_2 and V_3, we have $X_{21} = X_1 = 0$; $k_3 = k_4 = \infty$ and

(8.1.40) $$E(V) = \sigma_3^2$$

Case 2. For $\beta_1 = \beta_2 = \infty$, i.e., when the three mean squares V_1, V_2 and V_3 are always pooled we have $X_{21} = X_1 = 1$; $k_3 = k_4 = 1$ and

$$E(V) = \frac{n_1 \sigma_1^2 + n_2 \sigma_2^2 + n_3 \sigma_3^2}{n_{123}} \tag{8.1.41}$$

Case 3. For $\beta_1 = \infty$, $\beta_2 = 0$, i.e., when only two mean squares V_2 and V_3 are pooled we have $X_{21} = X_1 = 1$, $k_3 = k_4 = \infty$, and

$$E(V) = \frac{n_2 \sigma_2^2 + n_3 \sigma_3^2}{n_{23}} \tag{8.1.42}$$

8.2 Mean Square Error of V.

In order to find the mean square error of V, it is necessary to find $E(V^2)$ and then to use the relation

$$MSE(V) = E(V^2) - 2 \sigma_3^2 E(V) + \sigma_3^4 \tag{8.2.1}$$

If we follow the same method as in the case of $E(V)$, we get

$$\frac{E(V^2)}{\sigma_3^4} = \frac{n_3+2}{n_3} + \frac{n_2(n_2+2)\varphi_{23}^2}{n_{23}^2} I_{X_{21}}\left(\frac{n_3}{2}, \frac{n_2}{2}+2\right)$$

$$+ \frac{2n_2n_3 \varphi_{23}}{n_{23}^2} I_{X_{21}}\left(\frac{n_3}{2}+1, \frac{n_2}{2}+1\right) - \tag{8.2.2}$$

$$- \frac{n_2(n_3+2)(n_2+2n_3)}{n_3\, n_{23}^2} I_{X_{21}}\left(\frac{n_3}{2}+2, \frac{n_2}{2}\right) -$$

(B.2.2)

$$- \frac{n_1(n_1+2n_{23})(n_{123}+2)}{n_{123}\, n_{23}^2\, k_5\, k_3^{n_3/2}\, k_4^{n_2/2}} \left[\sum_{i=0}^{n_1/2-1} \binom{n_1/2-1}{i} \right.$$

$$\frac{(-1)^i}{(\frac{n_{23}}{2}+i+2)k_3^i} \left(\sum_{j=0}^{i} \binom{i}{j} \left(\frac{k_3}{k_4}\right)^j \right.$$

$$\left(\frac{B_{X_1}(\frac{n_3}{2}+i-j+2, \frac{n_2}{2}+j)}{k_3^2} + \frac{2\varphi_{23}}{k_4 k_3} B_{X_1}\left(\frac{n_3}{2}+i-j+1, \frac{n_2}{2}+j+1\right) + \right.$$

$$\left.\left.\left. \frac{\varphi_{23}^2}{k_4^2} B_{X_1}\left(\frac{n_3}{2}+i-j, \frac{n_2}{2}+j+2\right) \right)\right)\right] +$$

$$\frac{\varphi_{13}^2(n_{123}+2)}{n_{123}\, k_5\, k_3^{n_3/2}\, k_4^{n_2/2}} \left[\sum_{i=0}^{n_1/2+1} \binom{n_1/2+1}{i} \right.$$

$$\left. \frac{(-1)^i}{(\frac{n_{23}}{2}+i)\, k_3^i} \left(\sum_{j=0}^{i} \binom{i}{j} \left(\frac{k_3}{k_4}\right)^j B_{X_1}\left(\frac{n_3}{2}+i-j, \frac{n_2}{2}+j\right) \right) \right]$$

$$+ \frac{2\varphi_{13}(n_{123}+2)}{n_{123}\, k_5\, k_3^{n_3/2}\, k_4^{n_2/2}} \left[\sum_{i=0}^{n_1/2} \binom{n_1/2}{i} \right.$$

$$\frac{(-1)^{i}}{\left(\frac{n_{23}}{2}+i+1\right)k_3^{\,i}}\left[\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^{j}\right[$$

(8.2.2)

$$\frac{B_{X_1}}{k_3}\left(\frac{n_3}{2}+i-j+1,\ \frac{n_2}{2}+j\right)+\frac{\varphi_{23}}{k_4}\,B_{X_1}\left(\frac{n_3}{2}+i-j,\ \frac{n_2}{2}+j+1\right)\Bigg]\Bigg]\Bigg]$$

and

$$\frac{MSE(V)}{\sigma_3^{\,4}}=\frac{2}{n_3}+\frac{n_2(n_2+2)}{n_{23}^{\,2}}\ \varphi_{23}^{\,2}\ I_{X_{21}}\left(\frac{n_3}{2},\ \frac{n_2}{2}+2\right)+$$

$$+\frac{2n_2\,\varphi_{23}}{n_{23}}\left|\frac{n_3}{n_{23}}\,I_{X_{21}}\left(\frac{n_3}{2}+1,\ \frac{n_2}{2}+1\right)-I_{X_{21}}\left(\frac{n_3}{2},\ \frac{n_2}{2}+1\right)\right|+$$

(8.2.3)

$$+\frac{n_2}{n_{23}}\left[\,2\,I_{X_{21}}\left(\frac{n_3}{2}+1,\ \frac{n_2}{2}\right)-\frac{(n_3+2)(n_2+2n_3)}{n_3\ n_{23}}\right.$$

$$\left.I_{X_{21}}\left(\frac{n_3}{2}+2,\ \frac{n_2}{2}\right)\right]+\frac{(n_{123}+2)}{n_{123}\ k_5\ k_3^{\,n_3/2}\ k_4^{\,n_2/2}}\Bigg|\Bigg[\varphi_{13}^{\,2}$$

$$\sum_{i=0}^{n_1/2+1}\binom{n_1/2+1}{i}\frac{(-1)^{i}}{\left(\frac{n_{23}}{2}+i\right)k_3^{\,i}}\left[\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^{j}\right.$$

$$\left.B_{X_1}\left(\frac{n_3}{2}+i-j,\ \frac{n_2}{2}+j\right)\right]\Bigg]+$$

$$2\varphi_{13}\Bigg\{\Bigg[\sum_{i=0}^{n_1/2}\binom{n_1/2}{i}\frac{(-1)^i}{(\frac{n_{23}}{2}+i+1)k_3^{\,i}}\Bigg(\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^j$$

$$\Big[\frac{B_{x_1}}{k_3}\left(\frac{n_3}{2}+i-j+1,\ \frac{n_2}{2}+j\right)+\frac{\varphi_{23}}{k_4}B_{x_1}\left(\frac{n_3}{2}+i-j,\frac{n_2}{2}+j+1\right)\Big]\Bigg)\Bigg]-$$

(8.2.3)

$$-\Bigg[\frac{n_{123}}{(n_{123}+2)}\sum_{i=0}^{n_1/2\ -1}\binom{n_1/2\ -1}{i}\frac{(-1)^i}{(\frac{n_{23}}{2}+i)k_3^{\,i}}\Bigg[\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^j$$

$$B_{x_1}\left(\frac{n_3}{2}+i-j,\ \frac{n_2}{2}+j\right)\Bigg]\Bigg]\Bigg\}+\frac{n_1}{n_{23}}\Bigg\{\Bigg[\frac{2n_{123}}{(n_{123}+2)}\sum_{i=0}^{n_1/2\ -1}$$

$$\binom{n_1/2\ -1}{i}\frac{(-1)^i}{(\frac{n_{23}}{2}+i+1)k_3^{\,i}}\Bigg(\sum_{j=0}^{i}\binom{i}{j}\left(\frac{k_3}{k_4}\right)^j$$

$$\Big[B_{x_1}\left(\frac{n_3}{2}+i-j+1,\ \frac{n_2}{2}+j\right)+\frac{\varphi_{23}}{k_4}B_{x_1}\left(\frac{n_3}{2}+i-j,\ \frac{n_2}{2}+j+1\right)\Big]\Bigg)\Bigg]-$$

$$\Bigg[\frac{(n_1+2n_{23})}{n_{23}}\sum_{i=0}^{n_1/2\ -1}\binom{n_1/2\ -1}{i}\frac{(-1)^i}{(\frac{n_{23}}{2}+i+2)k_3^{\,i}}$$

$$\left[\sum_{j=0}^{i} \binom{i}{j} \left(\frac{k_3}{k_4}\right)^j \left[\frac{1}{k_3^2} B_{X_1}\left(\frac{n_3}{2}+i-j+2, \frac{n_2}{2}+j\right) + \right.\right.$$

(8.2.3)

$$+ \frac{2\,\rho_{23}}{k_4\,k_3} B_{X_1}\left(\frac{n_3}{2}+i-j+1, \frac{n_2}{2}+j+1\right) +$$

$$\left.\left. + \frac{\rho_{23}^2}{k_4^2} B_{X_1}\left(\frac{n_3}{2}+i-j, \frac{n_2}{2}+j+2\right) \right]\right]$$

Special Cases .

Case 1. For $\beta_1 = \beta_2 = 0$, that is, when we never pool the mean squares V_1, V_2 and V_3, we get

(8.2.4) $$V(V) = \frac{2\,\sigma_3^4}{n_3} .$$

Case 2. For $\beta_1 = \beta_2 = \infty$, i.e., when the three mean squares V_1, V_2 and V_3 are always pooled, we get

(8.2.5) $$MSE(V) = 2\,\frac{n_1\,\sigma_1^4 + n_2\,\sigma_2^4 + n_3\,\sigma_3^4}{n_{123}^2} + \left(\frac{n_1\,\sigma_1^2 + n_2\,\sigma_2^2 - n_{12}\,\sigma_3^2}{n_{123}}\right)^2$$

Case 3. For $\beta_1 = \infty$, $\beta_2 = 0$, i.e., when only two mean squares V_2 and V_3 are pooled we get

$$(8.2.6) \qquad MSE(V) = 2 . \frac{n_2 \sigma_2^4 + n_3 \sigma_3^4}{n_{23}^2} + \frac{n_2^2 (\sigma_2^2 - \sigma_3^2)^2}{n_{23}^2}$$

8.3 Mathematical Results.

Result 8.3.1

For a given set of degrees of freedom and $\varphi_{13} = \varphi_{23} = 1$, the mean value of V expressed as a fraction of σ_3^2 is greater than $(1 - \alpha_1)(1 - \alpha_2)$.

Proof : For $\varphi_{13} = \varphi_{23} = 1$, the joint distribution of mean squares V_1, V_2 and V_3 given by (8.1.1) reduces to

$$(8.3.1) \qquad g(V_1, V_2, V_3) = C' \, V_1^{n_1/2 - 1} \, V_2^{n_2/2 - 1} \, V_3^{n_3/2 - 1}$$

$$\exp \left[- \frac{n_1 V_1 + n_2 V_2 + n_3 V_3}{2 \sigma_3^2} \right]$$

where

C' is a constant.

Let us apply the transformation

$$u_1 = \frac{n_3 V_3}{n_2 V_2}, \quad u_2 = \frac{n_2 V_2}{n_1 V_1}, \quad \text{and} \quad w = \frac{n_1 V_1}{2 \sigma_3^2}$$

in (8.3.1). Then the jacobian of transformation is

$\frac{(2 \sigma_3^2)^3}{n_1 n_2 n_3} u_2 w^2$ and the joint distribution of u_1, u_2 and w is given by

$$h(u_1, u_2, w) = C\, u_1^{n_3/2 \,-1}\, u_2^{n_{23}/2 \,-1}\, w^{n_{123}/2 \,-1} \tag{8.3.2}$$

$$\exp\left[- w\,(1 + u_2 + u_1 u_2 \,)\right] .$$

where

$$C = \frac{1}{\Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right) \Gamma\left(\frac{n_3}{2}\right)} \tag{8.3.3}$$

Now, the estimate V and the ranges of integration for the variables u_1 and u_2 are given as follows :

(i) $0 < u_1 < u_1^*$, $0 < u_2 < \frac{u_2^*}{1 + u_1}$,

and $V = \frac{2\, \sigma_3^{\,2}\, w(1 + u_2 + u_1 u_2)}{n_{123}}$

(ii) $0 < u_1 < u_1^*$, $\frac{u_2^*}{1 + u_1} \leq u_2 \leq \infty$

and $V = \frac{2\, \sigma_3^{\,2}\, u_2 w(1 + u_1)}{n_{23}}$

(iii) $u_1^* \leq u_1 \leq \infty$, $0 \leq u_2 \leq \infty$,

and $V = \frac{2\, \sigma_3^{\,2}\, u_1 u_2\, w}{n_3}$

where

$$u_1^* = \frac{n_3}{n_2}\, \beta_1 \quad , \quad u_2^* = \frac{n_{23}}{n_1}\, \beta_2 \, .$$

Let $E^*(V)$ denote the expected value of V. Then

(8.3.4) $$E^*(V) = E_1^* + E_2^* + E_3^*$$

where

$$E_1^* = E(V \,/\, (0 \le u_1 \le u_1^* \,,\; 0 \le u_2 \le \frac{u_2^*}{1+u_1}))$$

$$P(\,0 \le u_1 \le u_1^*,\, 0 \le u_2 \le \frac{u_2^*}{1+u_1}\,)$$

$$E_2^* = E(V \,/\, (0 \le u_1 \le u_1^*,\, \frac{u_2^*}{1+u_1} \le u_2 \le \infty \quad)$$

$$P(0 \le u_1 \le u_1^*,\, \frac{u_2^*}{1+u_1} \le u_2 \le \infty \quad)$$

$$E_3^* = E(V \,/\, (u_1^* \le u_1 \le \infty \quad ,\, 0 \le u_2 \le \infty \quad)$$

$$P(u_1^* \le u_1 \le \infty \quad ,\, 0 \le u_2 \le \infty \quad)$$

Now

(8.3.5) $$E_1^* = 2\,\sigma_3^2\, C \int_{u_1=0}^{u_1^*} \int_{u_2=0}^{\frac{u_2^*}{1+u_1}} \int_{w=0}^{\infty} (1+u_2+u_1u_2)\, u_1^{n_3/2\,-1} . u_2^{n_{23}/2\,-1}\, w^{n_{23}/2} \exp\left[-\,w(1+u_2+u_1u_2)\right] dw\,du_1\,du_2$$

Integrating out w as a gamma variate, we obtain

$$(8.3.6) \qquad E_1^* = K \; \sigma_3^2 \int_0^{u_1^*} \int_0^{\frac{u_2^*}{1+u_1}} \frac{u_1^{n_3/2-1} \; u_2^{n_{23}/2-1} \; du_2 du_1}{(1+u_2+u_1u_2)^{n_{123}/2}}$$

where

$$(8.3.7) \qquad K = \left[B\left(\frac{n_3}{2}, \frac{n_2}{2}\right) B\left(\frac{n_{23}}{2}, \frac{n_1}{2}\right) \right]^{-1}$$

If we apply the transformation

$$u_1 = w_1$$

$$u_2 = \frac{w_2}{1+w_1}$$

in (8.3.6), we get after simplification

$$(8.3.8) \qquad E_1^* = \sigma_3^2 \left[\frac{1}{B\left(\frac{n_3}{2}, \frac{n_2}{2}\right)} \int_0^{u_1^*} \frac{w_1^{n_3/2-1} \; dw_1}{(1+w_1)^{n_{23}/2}} \right]$$

$$\left[\frac{1}{B\left(\frac{n_{23}}{2}, \frac{n_1}{2}\right)} \int_0^{u_2^*} \frac{w_2^{n_{23}/2-1} \; dw_2}{(1+w_2)^{n_{123}/2}} \right]$$

Using the relation between the F-distribution and the incomplete beta function, (8.3.8) may be written as

$$(8.3.9) \qquad E_1^* = (1-\alpha_1)(1-\alpha_2) \; \sigma_3^2$$

Since B_2^* and B_3^* are necessarily non-negative, we have from (8.3.4)

$$\frac{B^*(V)}{\sigma_3^2} > (1 - \alpha_1)(1 - \alpha_2) \quad (8.3.10)$$

<u>Result 8.3.2</u>

For $\varphi_{13} = \varphi_{23} = 1$; $\alpha_1 = \alpha_2 = \alpha$, say , the bias expressed as a fraction of σ_3^2 is greater than $\alpha(\alpha - 2)$.

<u>Proof</u> . From (8.3.10) the result follows immediately.