

CHAPTER 2

DYNAMIC PROGRAMMING AND RISK FUNCTIONS

2.1 INTRODUCTION

In recent years lot of work has been carried out dealing with processes in which the change in state from one stage to another stage is described in a stochastic manner, the parameters and the form of which are assumed to be known, possibly in terms of the decisions as well as the uncontrolled environment. This is the usual statistical approach to the problem. However, there are a large class of problems for which stochastic descriptions of the process are still valid, but whose form or parameters are unknown. We shall consider here this subclass of problem, namely, the process in which the parameters themselves are subject to stochastic laws and they can be modified as the information is gathered from stage to stage. These types of problems are usually known as 'adaptive programming problems' (Bellman [13], Marschak [69], White [94]), which are of mixed learning and optimizing type problems, and each action in the sequence is determined in two steps of data processing, viz., (i) re-estimating the probability distribution a posteriori, and (ii) computing the action on the basis of the new estimate.

Suppose that one wishes to control a certain system for which some measure of performance exists. This may depend upon the decisions made and the conditions assumed throughout the relevant life of the system. At any time, the condition of the system may be described by two sets of

variables, namely \underline{s} and \underline{x}_n^1 , where \underline{x}_n may be a set of n observations made on certain variables, not subject to the control of the decision maker, except that the variables to be chosen and \underline{s} represents all other necessary variables required so as to describe the condition of the system for the purpose of developing a model. For example, if we consider an inventory system \underline{s} and \underline{x}_n ($\underline{s} = [\underline{s}]$, $\underline{x}_n = [\underline{x}_n]$) represents the present inventory level and the previous period demand quantity.

It is assumed that the variable \underline{x}_n follows a certain probabilistic law. The problems which we consider here are those in which the form of the distribution of \underline{x}_n is known, but, within this form, there is an unknown parameter θ (in general, this can be a vector) subject to a probability density function $G_0(\theta)$, initially. This probability density function of the parameter θ is known as an a priori distribution. The distribution function for \underline{x}_n , given θ , is then $F_n(\underline{x}_n | \theta)$. Once we know the prior distribution of the parameter, θ , and the variable \underline{x}_n , we can determine the posterior distributions (Raiffa and Schlaifer [79]) for both the parameter, θ , and the next observation, \underline{x}_{n+1} , according to Bayes theorem. Thus, we have

$$dG_n(\theta | \underline{x}_n) = \frac{dG_0(\theta) dF_n(\underline{x}_n | \theta)}{\int_{\theta} dG_0(\theta) dF_n(\underline{x}_n | \theta)} \quad (2.1)$$

$$dF_n(\underline{x}_{n+1} | \underline{x}_n) = \frac{\int_{\theta} dG_0(\theta) dF_{n+1}(\underline{x}_n, \underline{x}_{n+1} | \theta)}{\int_{\theta} dG_0(\theta) dF_n(\underline{x}_n | \theta)} \quad (2.2)$$

¹ Letters with lower bars represent vectors.

We can see from (2.1) that the posterior distribution of the parameter θ is proportional to the product of the known density of \underline{x}_n and the prior distribution of θ .

The other main requirement now is that of specifying the preference between the policies and our aim is to choose that policy which will give rise to the maximum expected value or the minimum expected loss of the costs generated by the policy under the given environment conditions.

2.2 SEQUENTIAL DECISION FUNCTION APPROACH

Wald [90] has considered the following decision problem. A decision function ϑ is a rule for deciding, at each stage of the process, whether to terminate (and hence to choose one of the terminal acts) or to go on and take further observations, and if so, which one to take. In order to find the best decision now (i.e., in particular whether to go on further or not), it is necessary to know the best decision in the future.

If θ is the true state of the nature whose prior distribution function is $G_0(\theta)$, then there exists a risk function $r(\theta, \vartheta)$ which represents the expected loss (or gain), which can include the sampling costs and the terminal losses (or gains), for each decision function ϑ . Then the over-all risk for a particular decision function ϑ is given by

$$r(\vartheta) = \int_{\theta} r(\theta, \vartheta) dG_0(\theta) \quad (2.3)$$

A decision function ϑ^* is said to be admissible (Wald [90])

if there exists no other decision function ω which is uniformly better than ω^* , i.e., if there exists no decision function ω satisfying

$$r(\theta, \omega^*) \geq r(\theta, \omega) \quad (2.4)$$

for all θ and any ω , and

$$r(\theta, \omega^*) < r(\theta, \omega) \quad (2.5)$$

for at least one θ and any ω , and is essentially concerned with the determination of complete classes of admissible decision functions in the sense that if ω is any decision function not in the class, then there is a ω^* in the class such that $r(\theta, \omega^*) \leq r(\theta, \omega)$ for all θ .

DeGroot and Schlaifer [79] have studied these problems in which the prior distribution, $G_0(\theta)$, is known to exist, and a decision function which minimizes $r(\omega)$ is said to be a Bayes solution relative to $G_0(\theta)$.

2.2 DYNAMIC PROGRAMMING APPROACH

In decision theory problems, it is the idea of concentrating attention not on the optimum procedure itself but on the minimization of the expected risk obtained from the optimum procedure. Once the minimum risk has been found out, it is usually a simple problem to find out that procedure which will produce this risk. We shall use the dynamic programming technique to obtain a recurrence relation for the minimum risk so as to solve the problems analytically or numerically.

In most of the cases, it is possible to separate the risk function

into the sum of costs due to sampling and terminal risks so that we can apply the principle of optimality (Bellman [11]). Let $c(\underline{x}, s)$ be the observational (or sampling) cost of \underline{x} ($\underline{x} = (x_1, x_2, \dots, x_n)$) and s (where s specifies the manner in which \underline{x} is obtained), and if $r(\theta, \beta)$ is the terminal risk of taking action β when the true state is θ , we could compute $F(\underline{x}, s | \omega, \theta)$ (which is the distribution function for the observations \underline{x} , s prior to termination of the process, given ω and θ),

$$r(\omega) = \int_{\theta} \int_{\underline{x}} [c(\underline{x}, s) + r(\theta, \omega(\underline{x}, s))] dF(\underline{x}, s | \omega, \theta) dG_0(\theta) \quad (2.6)$$

However, once we have observed \underline{x} and the corresponding s , then we will be choosing our next decisions so as to optimize our future performance. If the new posterior distribution of θ , after observing \underline{x} and corresponding s , is dependent only on \underline{x} and s , and if this distribution is denoted by $G_n(\theta | \underline{x}, s)$, then using the principle of optimality, we get

$$v(\underline{x}, s) = \min \left\{ \begin{array}{l} \min_{\beta} \int_{\theta} r(\theta, \beta) dG_n(\theta | \underline{x}, s) \\ \min_{s^*} \int_{\theta} \int_{\underline{x}^*} [c(\underline{x}^*, s^*) - c(\underline{x}, s) + v(\underline{x}^*, s^*)] \\ \quad dG_n(\theta | \underline{x}, s) dF(\underline{x}^* | \underline{x}, s^*, \theta) \end{array} \right. \quad (2.7)$$

where $v(\underline{x}, s)$ is the expected risk, beginning with conditions \underline{x}, s and using an optimal policy, s may be equivalent to 'observations x_1, \dots, x_n have been taken; s^* might then be equivalent to 's plus observe x_{s+1}, \dots, x_s '. $F(\underline{x}^* | \underline{x}, s^*, \theta)$ is the distribution function

the x^* given x , s^* and s^* is the combination of s and the new sampling decision.

2.4 USE OF SUFFICIENT STATISTICS

A statistic Z_n (may be a vector also) is said to be a sufficient statistic (Wassan [91]) for the parameter θ , if

$$dF_n(x_1, \dots, x_n | \theta) = d\bar{\Phi}_n(Z_n | \theta) dX_n(x_1, \dots, x_n) \quad (2.8)$$

where X_n does not depend upon the parameter θ .

Sufficient statistics play an important role in dynamic programming approach and sometimes, it helps to reduce the dimensionality of the problem. In cases where the observations are independently distributed, when, if a sufficient statistics exists, it is possible to use this to represent the complete set of observations in the functional equation derived by the dynamic programming technique, and hence this may reduce the dimensionality problem to a considerable extent.

If the observations are independently distributed, then, from (2.1), (2.2) and (2.8), the posterior distributions of the parameter θ and the next observation x_{n+1} given x_1, x_2, \dots, x_n , respectively, are

$$dG_n(\theta | x_1, \dots, x_n) = \frac{d\bar{\Phi}_n(Z_n, \theta) dX_n(x_1, \dots, x_n) dG_0(\theta)}{\int_{\theta} d\bar{\Phi}_n(Z_n, \theta) dX_n(x_1, \dots, x_n) dG_0(\theta)} \quad (2.9)$$

$$dF_n(x_{n+1} | x_1, \dots, x_n) = \int_{\theta} dF_{n+1}(x_{n+1} | \theta) dG_n(\theta | x_1, \dots, x_n) \quad (2.10)$$

since $\phi_n(x_1, \dots, x_n)$ is independent of θ , (2.9) and (2.10) reduces to

$$dG_n(\theta | x_1, \dots, x_n) = \frac{dG_0(\theta) d\Phi_n(z_n, \theta)}{\int_{\Theta} d\Phi_n(z_n, \theta) dG_0(\theta)} \quad (2.11)$$

$$dF_n(x_{n+1} | x_1, \dots, x_n) = \frac{\int_{\Theta} d\Phi_n(z_n, \theta) dF_{n+1}(x_{n+1} | \theta) dG_0(\theta)}{\int_{\Theta} d\Phi_n(z_n, \theta) dG_0(\theta)}. \quad (2.12)$$

so z_n is the sufficient statistic for θ .

Let $A(\theta)$ be a real valued parametric function of the parameter

and let

$$L [g(x), A(\theta)] \quad (2.13)$$

be the loss incurred when $A(\theta)$ is estimated by a function $g(x)$ of the observations. We call L as the loss function.

A Bayesian estimator of $A(\theta)$ with respect to the loss function is that statistic $g^*(x)$ which, for a given x , minimizes the expected loss

$$E [L | x] = \int_{\Theta} L [g(x), A(\theta)] dG_n(\theta | x) \quad (2.14)$$

if it exists, where Θ is the parametric space.

If s and n are sufficient statistics for the dynamic programming formulation, then the functional equation given by (2.7) can be written as

$$v(s, n) = \min \left\{ \begin{array}{l} \min_{g(x)} \left\{ \int_{\Theta} L [g(x), A(\theta)] dG_n(\theta | s, n) \right\} \\ c + \int_x dF_n(x_{n+1} | s, n) f(s+x, n+1) \end{array} \right. \quad (2.15)$$

where $f(s, n)$ is the expected minimum risk and c is the unit sample cost.

2.5 QUADRATIC LOSS FUNCTION

Generally, in problems involving estimation of a real parameter θ , the loss function L is specified as a quadratic function (Robbins [81] and Maritz [65, 66]) so that

$$L[g(x), \theta] = k [g(x) - \theta]^2 \quad (2.16)$$

where k is a positive constant and $g(x)$, a function of x , is an estimator of θ .

Thus, for the quadratic loss function given by (2.16), the functional equation (2.15) can be written as

$$f(s, n) = \min \left\{ \begin{array}{l} \min_{g(x)} \left\{ \int k [g(x) - \theta]^2 dG_n(\theta | s, n) \right\} \\ c + \int_{\mathcal{X}} dx'_n(x_{n+1} | s, n) f(s+x, n+1) \end{array} \right. \quad (2.17)$$

Now let us suppose that x is an observation, which may possibly be a random vector, whose conditional probability density function when $\theta = \theta$ is $dP(\cdot | \theta)$. As usual, let $dG_0(\theta)$ denote the prior probability density function of θ and let $dG_n(\cdot | x)$ denote the posterior probability density function of θ when $x = x$. The Bayes estimator g^* and the Bayes risk L^* for the quadratic loss function given by (2.16) can be stated (DeGroot [33]) as follows: For any observed $x = x$, the Bayes estimator is

$$g^*(x) = E(\theta | x) \quad (2.18)$$

where $E(\theta | x)$ is the mean of the posterior distribution of the parameter θ which can be found to be a function of the sufficient statistics.

Furthermore, after the value x has been observed and the estimate $g^*(x)$ has been chosen, the risk is $k \cdot \text{Var}(\theta | x)$ where $\text{Var}(\theta | x)$ is the variance of the posterior distribution of θ . Hence, Bayes risk is

defined by the equation

$$L^*(x) = k \cdot E \left\{ \text{Var}(\theta | x) \right\} \quad (2.19)$$

The expectation in (2.19) is calculated with respect to the marginal probability function p_G of x where

$$p_G(x) = \int dF(x | \theta) dQ_0(\theta) \quad (2.20)$$

2.6 PARTICULAR CASES

We shall consider some a priori distributions for the random variable and for the unknown parameter and obtain the following useful results.

2.6.1 Expected Risk for the Normal Distribution

Let x_1, x_2, \dots, x_n be a random sample drawn from a normal population with mean μ and unit variance, where the value of the parameter μ is unknown. However, we shall assume that the a priori distribution of μ is also normal with known mean μ_0 and unit variance. So, we have

$$dF(x | \mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx \quad (2.21)$$

$$dG_0(\alpha) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\alpha - \mu)^2}{2}} d\alpha \quad (2.22)$$

Let $s = \left(\sum_{i=1}^n x_i \right) / n$ be the sufficient statistic for the

unknown parameter α . Then, the a posteriori distribution of α given

s and n (using (2.11)) is

$$dG_n(\alpha | s, n) = \frac{e^{-\frac{(\alpha - \mu)^2}{2}} e^{-\frac{n}{2}(s - \alpha)^2} d\alpha}{\int_{-\infty}^{\infty} e^{-\frac{n}{2}(s - \alpha)^2} e^{-\frac{(\alpha - \mu)^2}{2}} d\alpha}$$

Simplifying (given in appendix), we get

$$dG_n(\alpha | s, n) = \sqrt{\frac{n+1}{2\pi}} e^{-\frac{1}{2}(n+1) \left[\alpha - \frac{ns + \mu}{n+1} \right]^2} d\alpha \quad (2.23)$$

The a posteriori distribution of the next observation x_{n+1} given

s, \dots, x_n (using (2.12)) is

$$dG_n(x_{n+1} | s, n) = \frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{n}{2}(s - \alpha)^2} e^{-\frac{1}{2}(x_{n+1} - \alpha)^2} e^{-\frac{1}{2}(\alpha - \mu)^2} d\alpha}{\int_{-\infty}^{\infty} e^{-\frac{n}{2}(s - \alpha)^2} e^{-\frac{1}{2}(\alpha - \mu)^2} d\alpha} dx_{n+1}$$

Simplifying (given in Appendix), we obtain

$$dG_n(x_{n+1} | s, n) = \frac{1}{\sqrt{2\pi}} \int \frac{n+1}{n+2} e^{-\frac{n+1}{2(n+2)} \left[x_{n+1} - \frac{ns + \mu}{n+1} \right]^2} dx_{n+1} \quad (2.24)$$

Hence the a posteriori distribution of the parameter α is normal

with mean $(ns + \mu)/(n+1)$ and variance $1/(n+1)$. Also, the a posteriori distribution of x_{n+1} is normal with mean $(ns + \mu)/(n+1)$ and variance $(n+2)/(n+1)$. Thus s, n are sufficient statistics for the dynamic programming formulation, and for the quadratic loss function considered in the functional equation (2.17), the Bayes estimator g^* as defined in section 2.5 is given by

$$g^*(x_1, \dots, x_n) = \frac{ns + \mu}{n+1} \quad (2.25)$$

and since the variance of the a posteriori distribution of the parameter is independent of s , the Bayes risk (as given in equation (2.19)) is

$$L^*(x) = \frac{k}{n+1} \quad (2.26)$$

Hence, if $k/(n+1) \leq c$, the optimal solution is to take no further observation and to choose $g(x) = (ns + \mu)/(n+1)$ and $f(s, n) = k/(n+1)$, which is independent of s . Let N be such that

$$\frac{k}{N+1} \leq c < \frac{k}{N} \quad (2.27)$$

Then

$$f(s, n) = f(n) = \frac{k}{n+1} \quad \text{for } n \geq N \quad (2.28)$$

and

$$f(n) = \min \left\{ \begin{array}{l} \frac{k}{n+1} \\ c + f(n+1) \end{array} \right\} \quad \text{for } n < N \quad (2.29)$$

Thus, the optimal solution is of the form: take a sample of size n , determine the corresponding s and take $g^*(x) = (ns + \mu)/(n+1)$.

Table 2.1 below gives the decision procedure for specific values of k and c .

Let $k = 8$ and $c = 1$. Then

$$\frac{k}{N+1} \leq c < \frac{k}{N} \quad \text{gives} \quad N = 7.$$

Table 2.1 Decision procedure for normal distribution

I : $c \cdot f(n+1)$	II : $k/(n+1)$	$\min(I, II)$	Decision
1.0	1.0	1.0	T
2.0	1.14	1.14	T
2.14	1.33	1.33	T
2.33	1.60	1.60	T
2.60	2.00	2.00	T
3.00	2.67	2.67	T
3.67	4.00	3.67	C
5.00	8.00	5.00	C

T = Terminate and take $g^*(x) = \frac{ns + x}{n+1}$

C = Continue further with a sample.

6.2 Expected Risk for the Exponential Distribution

We shall now suppose that x_1, x_2, \dots, x_n is a random sample drawn from an exponential distribution with the parameter θ , whose value is unknown.

$$\text{i.e.,} \quad dF(x | \theta) = \frac{1}{\theta} e^{-x/\theta} dx, \quad x \geq 0, \theta > 0 \quad (2.30)$$

Let the prior distribution of θ be of the form

$$\begin{aligned} dG_0(\theta) &\propto \frac{1}{\theta^{\nu+1}} e^{-u/\theta} d\theta \\ &= \frac{1}{\Gamma(\nu)} \frac{\theta^{-\nu}}{\theta^{\nu+1}} d\theta \end{aligned} \quad (2.31)$$

where u and ν are known constants. The prior density of the form (2.31) is known as inverted gamma density function.

Let $s = \sum_{i=1}^n x_i$ be the sufficient statistic for the parameter θ .

Then s has the gamma distribution with n degrees of freedom.

The posterior distribution of the parameter θ is given by

$$dG_n(\theta | s, n) = \frac{\frac{e^{-u/\theta}}{\theta^{\nu+1}} \frac{1}{\theta^n} e^{-s/\theta} d\theta}{\int_0^\infty \frac{1}{\theta^n} e^{-s/\theta} \frac{1}{\theta^{\nu+1}} e^{-u/\theta} d\theta}$$

Simplifying (given in Appendix), we get

$$dG_n(\theta | s, n) = \frac{(s+u)^{n+\nu}}{\Gamma(n+\nu)} \frac{e^{-(s+u)/\theta}}{\theta^{n+\nu+1}} d\theta \quad (2.32)$$

This distribution also belongs to the inverted gamma family.

The posterior distribution of x_{n+1} given x_1, \dots, x_n is

$$dF_n(x_{n+1} | s, n) = \frac{\int_{\theta} \frac{1}{\theta^n} e^{-s/\theta} \frac{1}{\theta} e^{-x_{n+1}/\theta} \frac{1}{\theta^{n+1}} e^{-x_{n+1}/\theta} d\theta}{\int_{\theta} \frac{1}{\theta^n} e^{-s/\theta} \frac{1}{\theta^{n+1}} e^{-x_{n+1}/\theta} d\theta} dx_{n+1}$$

Simplifying (given in Appendix), we have

$$dF_n(x_{n+1} | s, n) = \frac{\Gamma(n+\nu+1)}{\Gamma(n+\nu)} \frac{1}{s^{+\nu}} \frac{dx_{n+1}}{\left(1 + \frac{x_{n+1}}{s}\right)^{n+\nu+1}} \quad (2.33)$$

The mean and the variance of the posterior distribution of the parameter θ , respectively, are

$$\text{mean} = E(\theta | x) = \frac{s + \frac{x}{\nu}}{n+\nu-1} \quad (2.34)$$

$$\text{variance} = V(\theta | x) = \frac{\left(s + \frac{x}{\nu}\right)^2}{(n+\nu-1)^2 (n+\nu-2)} \quad (2.35)$$

Thus s, n are sufficient statistics for the dynamic programming formulation, and for the quadratic loss function considered in the functional equation (2.17), the Bayes estimator as defined in section 2.5 is given by

$$g^k(x_1, \dots, x_n) = \frac{s + \frac{x}{\nu}}{n+\nu-1} \quad (2.36)$$

Since $E(x_i | \theta) = \theta$, for $i = 1, \dots, n$, it follows that

$$g(x_1) = E\left\{ \frac{1}{\nu} [x_1 - \theta] \right\} = E(\theta) = \frac{\nu}{\nu-1} \quad (2.37)$$

and hence from (2.19), (2.35) and (2.37), the Bayes risk is

$$L(x) = \frac{k\mu^2}{(\nu-1)^2(n+\nu-2)} \quad (2.38)$$

Using (2.38) in the functional equation (2.17), we can conclude

and if

$$\frac{k\mu^2}{(\nu-1)^2(n+\nu-2)} \leq c \quad (2.39)$$

the optimal solution is to take no further observation and to choose

$$f(s, n) = (s + \mu) / (n + \nu - 1) \text{ and } f(s, n) = k\mu^2 / [(\nu-1)^2(n+\nu-2)].$$

Let N be such that $k\mu^2 / (\nu-1)^2(N+\nu-2) \leq c < k\mu^2 / (\nu-1)^2(N+\nu-3)$,

then, for $n \geq N$

$$f(s, n) = f(n) = \frac{k\mu^2}{(\nu-1)^2(n+\nu-2)} \quad (2.40)$$

and for $n < N$

$$f(n) = \min \left\{ \begin{array}{l} \frac{k\mu^2}{(\nu-1)^2(n+\nu-2)} \\ c + f(n+1) \end{array} \right\} \quad (2.41)$$

Table 2.2 gives the decision procedure for specific values of

μ , ν and c .

Let $k = 6$, $\mu = 1$, $\nu = 2$ and $c = 1$. Then

$$\frac{k\mu^2}{(\nu-1)^2(N+\nu-2)} \leq c < \frac{k\mu^2}{(\nu-1)^2(N+\nu-3)} \text{ gives } N = 6$$

Table 2.2 Decision procedure for exponential distribution.

	I : $c \cdot f(n+1)$	II : $k \mu^2 / (\nu-1)^2 (n+1-2)$	$\min(I, II)$	Decision
	≥ 1.0	1.0	1.0	T
	2.0	1.2	1.2	T
	2.2	1.5	1.5	T
	2.5	2.0	2.0	T
	3.0	3.0	3.0	T/C
	4.0	6.0	4.0	C
	7.0		7.0	C

T = Terminate and take $g^*(x) = \frac{s + \mu}{n + \nu - 1}$

C = Continue further with a sample.

6.3 Expected Risk for the Poisson Distribution

Let us assume that x_1, \dots, x_n is a random sample drawn from a Poisson distribution with unknown parameter λ . Let the prior distribution of the parameter λ be a gamma distribution with known parameters α and β .

Thus, we have

$$dF(x | \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad (2.42)$$

$$dG_0(\lambda) = \frac{\alpha(\alpha\lambda)^{\beta-1} e^{-\alpha\lambda}}{\Gamma(\beta)} d\lambda \quad (2.43)$$

Let $s = \sum_{i=1}^n x_i$ be the sufficient statistic for the unknown

parameter λ . Then the posterior distribution of the parameter λ given

is

$$dG_n(\lambda | s, n) = \frac{e^{-(n+\alpha)\lambda} \lambda^{s+\beta-1} d\lambda}{\int_{\lambda} e^{-(n+\alpha)\lambda} \lambda^{s+\beta-1} d\lambda}$$

simplifying (given in Appendix), we get

$$dG_n(\lambda | s, n) = \frac{(n+\alpha)^{s+\beta}}{\Gamma(s+\beta)} e^{-(n+\alpha)\lambda} \lambda^{s+\beta-1} d\lambda \quad (2.44)$$

Thus, the posterior distribution of the parameter λ given s also belongs to the gamma family. The posterior distribution of x_{n+1} given x_1, \dots, x_n

is given by

$$dF_n(x_{n+1} | s, n) = \frac{\frac{1}{x_{n+1}!} \int_{\lambda} e^{-(n+1+\alpha)\lambda} \lambda^{s+x_{n+1}+\beta-1} d\lambda}{\int_{\lambda} e^{-(n+\alpha)\lambda} \lambda^{s+\beta-1} d\lambda}$$

simplifying (given in Appendix), we get

$$dF_n(x_{n+1} | s, n) = \left(\frac{n+\alpha}{n+\alpha+1} \right)^{s+\beta} \frac{\Gamma(s+x_{n+1}+\beta)}{\Gamma(s+\beta)} \frac{1}{x_{n+1}!} \frac{1}{(n+\alpha+1)^{x_{n+1}}} \quad (2.45)$$

which is of the negative binomial probability density function.

The mean and the variance of the posterior distribution of the parameter λ are given by

$$\text{mean} = E(\lambda | x) = \frac{\beta + s}{\alpha + n} \quad (2.46)$$

$$\text{variance} = V(\lambda | x) = \frac{\beta + s}{(\alpha + n)^2} \quad (2.47)$$

respectively, so that for the quadratic loss function considered in section 2.5, the Bayes estimator g^* is given by

$$g^*(x_1, \dots, x_n) = \frac{\beta + s}{\alpha + n} \quad (2.48)$$

Since $E(x_i | \lambda) = \lambda$, for $i = 1, \dots, n$, it follows that

$$E(x_1) = E\{E(x_1 | \lambda)\} = E(\lambda) = \frac{\beta}{\alpha} \quad (2.49)$$

Hence from (2.19), (2.47) and (2.49), the Bayes risk is

$$L^*(x) = \frac{k\beta}{\alpha(\alpha + n)} \quad (2.50)$$

Thus s, n are sufficient statistics for the dynamic programming formulation. Using (2.50) in the functional equation (2.17), we have, if $k\beta / \alpha(\alpha + n) \leq c$, then the optimal solution is to take no further observation and to choose $g(x) = (\beta + s) / (\alpha + n)$ and $f(s, n) = k\beta / \alpha(\alpha + n)$.

If N is an integer such that

$$\frac{k\beta}{\alpha(\alpha + N)} \leq c < \frac{k\beta}{\alpha(\alpha + N-1)}$$

then, for $n \geq N$

$$f(s, n) = f(n) = \frac{k\beta}{\alpha(\alpha + n)} \quad (2.51)$$

and for $n < N$

$$f(n) = \min \left\{ \begin{array}{l} \frac{k\beta}{\alpha(\alpha + n)} \\ c + f(n+1) \end{array} \right\} \quad (2.52)$$

In table 2.3 below, we give a decision procedure for specific values

Let k, α, β and c .

Let $k = 9, \alpha = \beta = 1$ and $c = 1$. Then

$$\frac{k \lambda^k}{\alpha(\lambda + N)} \leq c < \frac{k \lambda^k}{\alpha(\lambda + N - 1)}$$

was $N = 8$.

Table 2.3 Decision procedure for Poisson Distribution.

$I : \alpha + f(n+1)$	$II : k \lambda^k / \alpha(\lambda + n)$	$\min(I, II)$	Decision
≥ 1.0	1.0	1.0	T
2.0	1.13	1.13	C
2.13	1.29	1.29	T
2.29	1.50	1.50	C
2.50	1.80	1.80	T
2.80	2.25	2.25	C
3.25	3.00	3.00	T
4.00	4.50	4.00	C
5.50	9.00	5.50	C

T = Terminate and take $g^*(x) = \frac{\lambda^x \cdot \alpha}{\alpha + n}$

C = Continue further with a sample.

2.6.4 Expected Risk for the Binomial Distribution

We shall now assume that x_1, x_2, \dots, x_n is a random sample drawn

from the binomial population

$$dF(x | \lambda) = \binom{N}{x} \lambda^x (1-\lambda)^{N-x} \quad (2.53)$$

where λ is the unknown parameter. Let the prior distribution of λ be a beta distribution, so that

$$dG_0(\lambda) = \frac{\lambda^{\alpha-1} (1-\lambda)^{\beta-1}}{B(\alpha, \beta)} d\lambda \quad (2.54)$$

Let $s = \sum_{i=1}^n x_i$ be the sufficient statistic for the parameter λ .

Then, the posterior distribution of λ given s is (using (2.11))

$$dG_n(\lambda | s, n) = \frac{\lambda^s (1-\lambda)^{Nn-s} \lambda^{\alpha-1} (1-\lambda)^{\beta-1} d\lambda}{\int \lambda^s (1-\lambda)^{Nn-s} \lambda^{\alpha-1} (1-\lambda)^{\beta-1} d\lambda}$$

Simplifying (given in Appendix), we obtain

$$dG_n(\lambda | s, n) = \frac{\lambda^{s+\alpha-1} (1-\lambda)^{Nn+\beta-s-1}}{B(\alpha+s, Nn+\beta-s)} d\lambda \quad (2.55)$$

which is also a beta - type distribution. The posterior distribution of

λ given x_1, \dots, x_n is (using (2.12))

$$dF_n(x_{n+1} | s, n) = \frac{\binom{N}{x_{n+1}} \lambda^{x_{n+1}} (1-\lambda)^{N-x_{n+1}} \lambda^{s+\alpha-1} (1-\lambda)^{Nn+\beta-s-1} d\lambda}{\int \lambda^s (1-\lambda)^{Nn-s} \lambda^{\alpha-1} (1-\lambda)^{\beta-1} d\lambda}$$

Simplifying (given in Appendix), we get

$$dF_n(x_{n+1} | s, n) = \binom{N}{x_{n+1}} \frac{B(s+x_{n+1} + \alpha, N(n+1) + \beta - s - x_{n+1})}{B(s+\alpha, Nn + \beta - s)} \quad (2.56)$$

The mean and the variance of the posterior distribution of the parameter

are

$$\text{mean} = E(\lambda | x) = \frac{\alpha + s}{Nn + \alpha + \beta} \quad (2.57)$$

$$\text{variance} = V(\lambda | x) = \frac{(\alpha + s)(Nn + \beta - s)}{(Nn + \alpha + \beta)^2 (Nn + \alpha + \beta + 1)} \quad (2.58)$$

respectively. Therefore, for the quadratic loss function considered in section 2.5, the Bayes estimator g^* is given by

$$g^*(x_1, \dots, x_n) = \frac{\alpha + s}{Nn + \alpha + \beta} \quad (2.59)$$

Since $p(x_i | \lambda) = N\lambda$, for $i = 1, \dots, n$, it follows that

$$E(x_i) = E \left[\int_0^1 x_i \lambda \right] = N E(\lambda) = \frac{N\alpha}{\alpha + \beta} \quad (2.60)$$

Thus using (2.60) in the functional equation (2.17), we conclude that

$$\frac{k\alpha^2}{(\alpha + \beta)^2 (Nn + \alpha + \beta + 1)} \leq c$$

The optimal solution is not to take any more observations and to choose

$$g(s, n) = (s + \alpha) / (Nn + \alpha + \beta) \text{ and } f(s, n) = k\alpha^2 / (\alpha + \beta)^2 (Nn + \alpha + \beta + 1).$$

If N' is an integer such that

$$\frac{k \alpha \beta}{(\alpha + \beta)^2 (N N' + \alpha + \beta + 1)} \leq c < \frac{k \alpha \beta}{(\alpha + \beta)^2 [N(N' - 1) + \alpha + \beta + 1]}$$

then, for $n \geq N'$

$$f(s, n) = f(n) = \frac{k \alpha \beta}{(\alpha + \beta)^2 (N n + \alpha + \beta + 1)} \quad (2.61)$$

or for $n < N'$

$$f(n) = \min \left\{ \begin{array}{l} \frac{k \alpha \beta}{(\alpha + \beta)^2 (N n + \alpha + \beta + 1)} \\ c + f(n + 1) \end{array} \right\} \quad (2.62)$$

It is possible to obtain a set of decision procedures for specific values of k, c, α, β and N .

A FORTRAN program is included in Appendix so that the numerical solutions obtained here can also be obtained with the help of a computer. The use of computer will help us to obtain these results for different sets of values of constants and parameters. The program has been tested and run on IBM 1620 computer system attached to the University.