CHAPTER 3

DYNAMIC PROGRAMMING TECHNIQUE AND OPTIMIZATION PROBLEMS

SECTION I : AN OPTIMAL DECISION PROCEDURE USING DYNAMIC PROGRAMMING TECHNIQUE

3.1.1 INTRODUCTION

We shall consider a situation where customers deposit a certain amount of money every month with a trading concern continuously over a duration of time, say N months. At the end of N/2 (N is considered to be even) months, the trading concern will issue credit cards to the customers worth of double the amount of what they have actually deposited till then, under the assumption that they (customers) would continue to pay their monthly subscription up to the end of N months, so that the customers can purchase goods from an authorised dealer equivalent to that amount. There is a certain amount of loss due to the possible discontinuation of payment by the customers soon after they are being issued with credit cards for the purchase of articles (i.e., soon after the purchase of articles from the dealer). However, the dealer would pay a certain percentage of commission to the trading concern as remuneration towards the new business offered to the dealer by the trading concern. There may also be some additional incidental expences towards establishment, etc., incurred by the trading concern.

Because of these physical characteristics of the process, it may not be very economical (i.e., the loss incurred by the trading concern may be very high as compared to the commission allowed by the dealer) to continue the process for a length of N months and a decision has to be taken as to when the process should be revised. If the revision costs were zero, then one is inclined to revise the process more frequently (i.e., there may be many cycles of shorter duration). However, since there is a certain set up cost per cycle, there results in cycles of longer duration.

§ 1.2 FORMULATION OF THE PROBLEM

Let us suppose that, at the beginning of the process, we know the state of the system, i.e., the values of the parameters α and β (say) which will determine the over-all loss if we continue the process over the duration of n , say, months where α may be the factor contributed by the loss incurred due to the possible discontinuation of payment during the latter part of the planning horizon and β may be the factor contributed by the gain due to the percentage of commission allowed by the dealer. Let us now denote the loss incurred during this cycle consisting of n months by the function $l(\alpha, \beta, n)$.

Our problem is to minimise the expected total loss when the process is continued over a duration of N months. Once we know the parameters α and β and the revision policy we intend to adopt (which tells us when to revise the process when once α and β are known), the expected over-all loss over the next N months will be a function of α, β and N only. Let us denote this policy, when the optimal decision procedure is used, by the function $f_N(\alpha, \beta)$, the immediate loss (when the process is carried over n months) being $l(\alpha, \beta, n)$.

3.1.3 DYNAMIC PROGRAMMING APPROACH

The loss over the remaining duration of $(N - n)$ months (i.e., at the end of the first cycle) will depend upon the new values of the parameters α, β, when the process is revised, and the duration, $N-n$, of the process. Thus, using the principle of optimality (Bellman [11]), the optimal loss over the remaining duration of $(N-n)$ months would be

$$f_{N-n}(\gamma, \omega) \tag{3.1}$$

where γ and ω are the resultant parameters at the end of n months (i.e., the parameters at the start of the next cycle corresponding to the original parameters α and β respectively).

Since we do not know the exact form of the new parameters γ and ω, we shall have to estimate the values of these parameters on the basis of the past information. That is, we assume the joint probability density function of the parameters γ and ω is of the form $dG(\gamma, \omega)$.

Therefore, the expected loss over the remaining duration of $(N-n)$ months would be (using 3.1)

$$\int_{\gamma}\int_{\omega} f_{N-n}(\gamma, \omega)\, dG(\gamma, \omega) \tag{3.2}$$

Hence, using the principle of optimality, the recurrence relation for the over-all expected minimum loss for the N month time period can be expressed as

$$f_N(\alpha, \beta) = \underset{0 \le n \le N}{\text{Min}} \left[l(\alpha, \beta, n) + \int_{\gamma}\int_{\omega} f_{N-n}(\gamma, \omega)\, dG(\gamma, \omega) \right] \tag{3.3}$$

we can obtain the solution of (3.3) as follows :

Generally, one is interested in finding an optimal policy which will minimize the over-all loss when the process is continued over a long duration of time, i.e., when N tends to infinity. Under certain conditions (Howard [51], White [94]), $f_N(\alpha, \beta)$ takes the form [1]

$$f_N(\alpha, \beta) = Ng + f(\alpha, \beta) \tag{3.4}$$

as $N \to \infty$ and where g is the average loss per unit time.

Using (3.4) and for large N, we can write the equation (3.3) as

$$f(\alpha, \beta) = \min_{n \geq 0} \left\{ l(\alpha, \beta, n) - ng + \int_{\gamma}\int_{\vartheta} f(\gamma, \vartheta)\, dG(\gamma, \vartheta) \right\} \tag{3.5}$$

Substituting

$$\int_{\gamma}\int_{\vartheta} f(\gamma, \vartheta)\, dG(\gamma, \vartheta) = u \tag{3.6}$$

we can write (3.5) as

$$f(\alpha, \beta) = \min_{n \geq 0} \left\{ l(\alpha, \beta, n) - ng + u \right\} \tag{3.7}$$

If the loss over the initial cycle of duration of n months is of the form

$$l(\alpha, \beta, n) = \alpha n^2 - \beta n + m \tag{3.8}$$

---

[1] $f_N(\alpha, \beta) = Ng + f(\alpha, \beta) + \epsilon_N(\alpha, \beta)$ where $\epsilon_N(\alpha, \beta) \to 0$ as $N \to \infty$ and $|\epsilon_N(\alpha, \beta)|$ need only be uniformly bounded for all α, β and N.

where m is a constant (this factor may be the fixed set-up cost) then, substituting (3.8) in (3.7), we have

$$f(\alpha, \beta) = \underset{n \geq 0}{\text{Min}} \left\{ \alpha n^2 - (\beta + g) n + \mu + m \right\}$$

$$= \underset{n \geq 0}{\text{Min}} \left\{ \alpha \left[n - \frac{\beta + g}{2\alpha} \right]^2 + \mu + m - \frac{(\beta + g)^2}{4\alpha} \right\} \qquad (3.9)$$

From (3.9), it can be seen that the minimum value of $f(\alpha, \beta)$ is attained when

$$n = \frac{\beta + g}{2\alpha} \qquad (3.10)$$

and the minimum value of the over-all loss is

$$f(\alpha, \beta) = \mu + m - \frac{(\beta + g)^2}{4\alpha} \qquad (3.11)$$

Using this result in the definition of μ (i.e., using (3.11) in (3.5)), we have

$$\mu = \int_{\gamma}\int_{\omega} \left[\mu + m - \frac{(\omega + g)^2}{4\gamma} \right] dG(\gamma, \omega)$$

Simplifying, we get

$$4m = \int_{\gamma}\int_{\omega} \frac{(\omega + g)^2}{\gamma} \, dG(\gamma, \omega) \qquad (3.12)$$

Once we have solved (3.12) for the value of g, we can obtain the value of n from (3.10).

3.4 SOLUTION OF g WHEN α AND β ARE INDEPENDENTLY DISTRIBUTED

Let the two parameters α and β be independently distributed so that

... joint probability density function $G(\gamma, \omega)$ (the resultant parameters ... and ... are also independently distributed) can be expressed as

$$dG(\gamma, \omega) = d\phi(\gamma)\, d\theta(\omega) \tag{3.13}$$

Substituting (3.13) in (3.12), we get

$$\ldots = \left[\int_{\gamma} \frac{1}{\gamma}\, d\phi(\gamma) \right] \left[\int_{\omega} (\omega + g)^2\, d\theta(\omega) \right] \tag{3.14}$$

...ting

$$k = \int_{\gamma} \frac{1}{\gamma}\, d\phi(\gamma)$$

... (3.14), we can write

$$\int_{\omega} (\omega + g)^2\, d\theta(\omega) = 4mk^{-1} \tag{3.15}$$

... PARTICULAR CASES

... shall now derive the explicit solutions for gamma, exponential, ...rmal and beta distributions.

... - gamma and ... - gamma

Let

$$d\phi(\gamma) = \frac{b^a \gamma^{a-1} \exp\{-b\gamma\}\, d\gamma}{\Gamma(a)}, \text{ for } \gamma > 0;\ a, b > 0$$

$$d\theta(\omega) = \frac{c^d \omega^{d-1} \exp\{-c\omega\}\, d\omega}{\Gamma(d)}, \text{ for } \omega > 0;\ c, d > 0$$

... the density functions of γ and ω respectively. Then we have

$$k = \frac{b^a}{\Gamma(a)} \int_0^{\infty} \frac{1}{\gamma} \gamma^{a-1} \exp\{-b\gamma\}\, d\gamma$$

Simplifying, we get

$$k = \frac{b}{a-1} \quad , \quad a > 1 \tag{3.16}$$

Also

$$\int_0^{\infty} (\omega + g)^2 \, d\theta(\omega) = \frac{c^d}{\Gamma(d)} \int_0^{\infty} (\omega + g)^2 \, \omega^{d-1} \exp\{-c\,\omega\} \, d\omega$$

On simplification, we have

$$\int_0^{\infty} (\omega + g)^2 \, d\theta(\omega) = \frac{d(d+1)}{c^2} + 2g\,\frac{d}{c} + g^2 \tag{3.17}$$

Using (3.16) and (3.17) in (3.15), we get

$$g^2 + 2g\,\frac{d}{c} + \frac{d(d+1)}{c^2} = \frac{4m(a-1)}{b} \tag{3.18}$$

(3.18) can be solved to obtain the value of g.

γ - gamma and δ - exponential

We shall now assume that the density function of ω is of the form

$$d\theta(\omega) = \lambda \exp\{-\lambda\omega\} d\omega \quad , \quad \text{for } \omega \geq 0, \ \lambda > 0$$

So that $\int_0^{\infty} (\omega + g)^2 \, d\theta(\omega) = \lambda \int_0^{\infty} (\omega + g)^2 \exp\{-\lambda w\} \, d\omega$

Simplifying, we get

$$\int_0^{\infty} (\omega + g)^2 \, d\theta(\omega) = g^2 + 2g\,\lambda^{-1} + 2\,\lambda^{-2} \tag{3.19}$$

Hence, from (3.16), (3.19) and (3.15), we get

$$g^2 + 2g\,\lambda^{-1} + 2\,\lambda^{-2} = \frac{4m(a-1)}{b} \tag{3.20}$$

which can be solved to obtain the value of g.

γ - gamma and ω - normal

We shall now assume that the random variable ω follows a normal distribution with zero mean and standard deviation σ so that

$$d\theta(\omega) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\omega^2/2\sigma^2\right) d\omega, \quad -\infty < \omega < \infty.$$

with $\theta(o)$ so small that negative values of ω are insignificant.

Then

$$\int_0^\infty (\omega + g)^2 d\theta(\omega) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^\infty (\omega + g)^2 \exp\left(-\omega^2/2\sigma^2\right) d\omega$$

so that, on simplification, we get

$$\int_0^\infty (\omega + g)^2 \, d\theta(\omega) = g^2 + 2g\sigma\sqrt{\frac{2}{\pi}} + \sigma^2 \tag{3.21}$$

Substituting (3.21) and (3.16) in (3.15), we get

$$g^2 + 2g\sigma\sqrt{\frac{2}{\pi}} + \sigma^2 = \frac{4a(a-1)}{b} \tag{3.22}$$

Hence we can solve (3.22) and obtain the value of g.

γ - beta and ω - beta

Let us now assume that the density functions of γ and ω belong to the beta family given by

$$d\rho(\gamma) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \gamma^{a-1} (1-\gamma)^{b-1} d\gamma, \quad \text{for } 0 < \gamma < 1,\ a, b > 0$$

$$d\theta(\vartheta) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} \vartheta^{c-1}(1-\vartheta)^{d-1} d\vartheta \quad , \text{ for } 0 < \vartheta < 1, \ c, \ d > 0$$

and

$$k = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{1}{\gamma} \gamma^{a-1} (1-\gamma)^{b-1} \, d\gamma$$

Simplifying, we get

$$k = \frac{a+b-1}{a-1} \quad , \quad a > 1 \tag{3.23}$$

and

$$\int (\vartheta + g)^2 \, d\theta(\vartheta) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} \int_0^1 (\vartheta + g)^2 \vartheta^{c-1} (1-\vartheta)^{d-1} d\vartheta$$

Simplifying, we obtain

$$\int (\vartheta + g)^2 \, d\theta(\vartheta) = g^2 + 2g \frac{c}{c+d} + \frac{c(c+1)}{(c+d)(c+d+1)} \tag{3.24}$$

Substituting (3.23) and (3.24) in (3.15), we obtain

$$g^2 + 2g \frac{c}{c+d} + \frac{c(c+1)}{(c+d)(c+d+1)} = \frac{4m(a-1)}{a+b-1} \tag{3.25}$$

which can be solved to obtain the value of g.

3.1.6 NUMERICAL SOLUTION

In the previous section, under 5.2, we have assumed that α follows a gamma type distribution with parameters a and b and β follows an exponential type distribution with parameter λ. In particular, let $a = 3$, $b = 1$ and $\lambda = 1$. Let $m = 10$, so that from (3.20) we obtain

$$g^2 + 2g + 2 = 80$$

olving this for g and taking the positive value (since we are interested in the average loss per unit time), we get $g = 7.89$. Then, for $\alpha = 0.5$ and $\beta = 2.5$, the optimal value of n (from (3.10)) is approximately equal to 10, so that the optimal decision policy is to collect the amount over a duration of 10 months initially, and then to carry on the procedure depending upon the new values of the parameter.

3.1.7 CONCLUSION

Here, one can suppose that α depends upon the probability of non-payment of money by the customers during the second - half of the cycle and β depends upon the amount of monthly deposit by the customers and the rate of commission allowed by the dealer to the trading concern. The constant a may be some fixed revision or set up cost.

With some modifications, the problem can also be viewed as a maximization problem and a solution to the problem can be obtained [92]. Such of these problems can arise when a manufacturer wants to introduce a new product to the market. To capture the market, initially, for a certain duration of time the manufacturer would like to give some incentive to the customers whereby he may incur a loss and thus he would like to know the optimum length of the time period so as to minimize his expected total loss. Viewed in another direction, in some situations, the decision maker would like to know the optimum number of persons (instead of months) to be grouped together at a time so as to give them some benefits on the purchases made by them so as to minimize the over-all expected loss. In such a situation, the use of optimization technique will help the decision maker to run the process most efficiently.

An ideal situation of this type has been described by Arnold Bennett [2] in his novel which can be summerised as follows : Many tradesman in the Five Towns had formed clubs so that their customers pay so much during every month to the tradesmen, who charged them nothing for keeping it, and at the end of the agreed number of months they purchased goods worth of the total amount deposited by them. Denry, a philanthropist of the Five Towns, wanted to start a new club and as a special inducement and to prove superior advantages to the ordinary clubs, wanted to allow his customers to spend their (customers') full nominal subscription to the club as soon as they have actually paid only half of it. Thus, for example, after paying fifty dollars (say, ten dollars a month) the customer could spend one hundred dollars in Denry's chosen shops and Denry would settle with the shops at once, while collecting the balance every month from the customers. These benefits to the customers were without any charge whatsoever to them. The factor that influences the decision is the loss incurred to Denry due to the possible discontinuation of payment by the customers during the later part of the planning horizon (i.e., after the purchase of articles from Denry's chosen shops). There is a certain amount of remuneration received by Denry from the shop-keepers to whom Denry would give new customers. They (shop-keepers) were to allow him a certain percentage of discount on all transactions affected through Denry. The problem faced to Denry was how long the collection at a time (i.e., the duration of the period) be made so as to minimise his total loss, taking into consideration the loss incurred due to the discontinuation of payment by the customers and the commission allowed by the shop-keepers. In situations of this type, the dynamic programming technique would give an useful solution.

SECTION II : OPTIMAL DECISION RULES FOR A STOCHASTIC INVESTMENT PROBLEM

2.1 INTRODUCTION

A stochastic investment problem arises when a decision maker with capital (cash) to invest is confronted by a sequence of investment opportunities and the amount of investment to any opportunity is specified by the probability distribution, the return on investment depending upon the amount invested. At a specified time, the decision maker, given the investment opportunities, must compare his present expected gain with his future estimated return and on some basis must decide which investment opportunity to accept. We shall develop here a method for optimal decision rules based on the dynamic programming technique (White [93]) for a class of such investment problems.

Many authors have attempted to determine the values of investment opportunities for the purpose of investment selection. Of these, Hirshleifer [48] gives the present value method and Lutz, et. al. [64] give the capital supply and demand method. These methods do not take into consideration either the future investment opportunities available to the decision maker or the total capital available for investment. Fisher [40] has studied this problem under the assumption that (i) the capital of the decision maker is fixed at some point in time and that his interest is in investing them over some future interval, and (ii) the capital of the decision maker is augmented throughout time and the decision maker's interest is to invest a stream of asseets. Cord [25] has given an approach for optimally selecting capital investmente with certain returns under the conditions of limited funds and a constraint on the maximum average variance allowed in the final investment package. This study is similar to Markovits's [68] work on portfolio selection. White [94] has studied this problem under the assumption that there are situations in which investment values, independent of present conditions and opportunities, do exist, and with in this class, there are others where the present worth valuation is correct. Further, he assumed that the system is closed, the arrival of opportunities are probabilistic and the income streams for each opportunity is also probabilistic and that the objective is to maximize the expected sum of liquid reserves plus the total cash invested to date after n periods of time. We shall consider here the following class of investment decision making problem. The capital (cash) of the decision maker is fixed at the beginning of the period and his interest is in investing the capital over an interval of time period so as to maximize the total return on investment. There may exist one or more investment opportunities and the amount invested to any of the opportunity is probabilistic, the return on investment depending upon the amount invested.

5.2 STRUCTURE OF THE PROBLEM

We shall assume that the decision maker is interested in maximizing the over-all returns due to investment of capital over a given interval of

... period. Let the decision-maker, at the beginning of the planning
...izon, have a certain amount of capital, say x units, for investment.
...ther, let us assume that he is faced with a sequence of investment
...ortunities and the amount invested to any opportunity is probabilistic,
... total amount thus invested being equal to the available capital. The
...turn associated with each opportunity is a function of the amount invested.
... the problem is one of devicing an optimal policy for deciding how
... to invest to the opportunities available in order to maximize the
...ected value of return.

.3 DYNAMIC PROGRAMMING APPROACH

1. Discrete formulation.

Let us define the following quantities :

$p_i(x, m)$ = the probability that m units are invested from a total available capital of x units to the i^{th} opportunity.

$p_i(x, o)$ = the probability that no amount is invested to the i^{th} opportunity when the capital available is x units.

$r_i(m)$ = the return from the i^{th} opportunity when m units are invested.

We shall define f(x) to be the expected value of the over-all return ... the available opportunities when a capital of x units are available ... investment.

Then, using the principle of optimality (Bellman [11]) , Thus

$$f(x) = \max_i \left[\sum_{m=o}^{x} r_i(m)\, p_i(x,m) + \sum_{m=o}^{x} f(x-m)\, p_i(x,m) \right] \qquad (3.26)$$

where $f(x-m)$ is the expected value of the return from the remaining opportunities when m units are invested to the i^{th} opportunity.

We shall assume that $p_i(x,o) \neq 1$ for any x and i, so that

$$\left[1 - p_i(x,o)\right] f(x) \geq \left[\sum_{m=o}^{x} r_i(m)\, p_i(x,m) + \sum_{m=1}^{x} f(x-m)\, p_i(x,m) \right]$$

so

$$f(x) \geq \left[\sum_{m=o}^{x} r_i(m)\, p_i(x,m) + \sum_{m=1}^{x} f(x-m)\, p_i(x,m) \right] \Big/ \left[1-p_i(x,o)\right] \qquad (3.27)$$

For some value of i we have the equality sign in the above equation, so that we get

$$f(x) = \max_i \left\{ \left[\sum_{m=o}^{x} r_i(m)\, p_i(x,m) + \sum_{m=1}^{x} f(x-m)\, p_i(x,m) \right] \Big/ \left[1-p_i(x,o)\right] \right\} \quad (3.28)$$

Initially, we assume that $f(o) = 0$ so that we can obtain $f(1)$, and then compute $f(x)$ inductively for all values of x.

Continuous formulation.

If x is very large the method of solution, as described above which requires that $f(x)$ be computed for all values of x, is time consuming and tedious. Sometimes, there is some advantage if we initially consider the process in continuous form, determine the degree of continuity and then use the properties of the solution to cut down the computation.

However, ultimately, all computations are to be carried out in discrete terms. We use Bellman's method [11] of successive approximations to solve the problem.

Let $r_i(m)$ be the value of the return from the i^{th} opportunity when m units are invested, and let $F_i(x,m)$ be the probability density function of the amount of investment to the i^{th} opportunity when, at the beginning of the operation, the decision-maker has a capital of x units for investment.

Defining $f(x)$ to be the expected value of return when the capital available for investment is x units and an optimal investment policy is used, then

$$f(x) = \max_i \left[\int_0^x r_i(m)dF_i(x,m) + \int_0^x f(x-m)dF_i(x,m) \right] \tag{3.29}$$

Following Bellman's method of successive approximations, we obtain the solution for (3.29). Let us define

$$f_1(x) = \max_i \left[\int_0^x r_i(m)\, dF_i(x,m) \right] \tag{3.30}$$

and for $n \geq 2$,

$$f_n(x) = \max_i \left[\int_0^x r_i(m)\, dF_i(x,m) + \int_0^x f_{n-1}(x-m)\, dF_i(x,m) \right] \tag{3.31}$$

We can now show that $f_n(x)$ is monotonically increasing function. From (3.31), we have

$$f_n(x) - f_{n-1}(x) \geq \min_i \left\{ \int_0^x \left[f_{n-1}(x-m) - f_{n-2}(x-m) \right] dF_i(x,m) \right\} \tag{3.32}$$

n = 2, we get

$$f_2(x) - f_1(x) \geq \min_i \left\{ \int_0^x f_1(x-m)\, dF_i(x,m) \right\} \tag{3.33}$$

ince the expression on the right hand side of (3.33) is greater than zero,

have

$$f_2(x) \geq f_1(x) \tag{3.34}$$

oceeding similarly and by the inductive method, we can show that

$$f_n(x) \geq f_{n-1}(x) \tag{3.35}$$

ence $f_n(x)$ is a monotonically increasing function. We can also show that (x) converges to $f(x)$ with the help of the method utlined in llman [11].

Let us assume that the distribution function is independent of n.

$$T(i,x,f) = \int_0^x r_i(m)\, dF_i(x,m) + \int_0^x f(x-m)\, dF_i(x,m) \tag{3.36}$$

hen, the equation (3.29) can be expressed as

$$f(x) = \max_i \left[T(i,x,f) \right] \tag{3.37}$$

each $n \geq 1$, let i_n be the value of i for which

$T(i,x,f_n)$ attains its maximum. We, then, have

$$f_{n+1} = T(i_{n+1}, x, f_n) \geq T(i_n, x, f_n)$$

$$f_n = T(i_n, x, f_{n-1}) \geq T(i_{n+1}, x, f_{n-1})$$

Combining these two inequalities, we can obtain

$$|f_{n+1} - f_n| \leq \max \{ |T(i_{n+1}, x, f_n) - T(i_{n+1}, x, f_{n-1})|, \; |T(i_n, x, f_{n-1}) - T(i_n, x, f_n)| \} \quad (3.38)$$

i.e.,

$$|f_{n+1} - f_n| \leq \max \left\{ \int_0^x |f_n(x-m) - f_{n-1}(x-m)| \, dF_1(x,m), \; \int_0^x |f_n(x-m) - f_{n-1}(x-m)| \, dF_1(x,m) \right\} \quad (3.39)$$

Therefore,

$$\max_x |f_{n+1}(x) - f_n(x)| \leq \max_x |f_n(x) - f_{n-1}(x)| \int_0^x dF_1(x,m) \quad (3.40)$$

Since $dF_1(x,m) \geq 0$ and $\int_0^x dF_1(x,m) \leq 1$

we have

$$\max_x |f_{n+1}(x) - f_n(x)| \leq \max_x |f_n(x) - f_{n-1}(x)| \quad (3.41)$$

Thus, the series $\sum_{n=0} (f_{n+1}(x) - f_n(x))$ converges uniformly in a finite interval for all $x \geq 0$, and thus $f_n(x)$ converges to $f(x)$ for all $x \geq 0$.

3.4 USE OF DISCOUNT FACTOR

We now give an alternative approach to the above investment problem using the discount factor. The problem is restated as follows :

At the beginning of the process, the decision - maker has x units capital for investment. He can either invest all the capital during the initial period or invest part of it during the first period and reserve the balance for investment during the succeeding periods. The problem is to maximize the total discounted return on investment, when the planning horizon consists of n periods of equal length. Let (i) the return during any period be assumed to be independent of the period under consideration but is a function of the amount invested so that $r(m)$ denotes the return during any period when m units are invested, and (ii) let $p(x,m)$ denote the probability that m units are invested from a total available capital of x units during any period. Hence $p(x,o)$ denotes the probability of not making any investment during any period when there remains x units for investment during the remaining periods. Thus, the amount of investment during any period is probabilistic and this probability of investment is independent of the period under consideration.

Let $f_n(x)$ be the maximum expected discounted return on investment of x units when an optimal decision making policy for investment is employed during the planning horizon of n periods. The function $f_n(x)$ will be the result of a complex process that includes the number of periods, the sequence of probability functions, the available amount of investment and the sequence of optimal decision rules employed. Thus using Bellman's principle of optimality, we have

$$f_n(x) = \max \left[\sum_{m=0}^{x} r(m)\, p(x,m) + \alpha \sum_{m=0}^{x} f_{n-1}(x-m)\, p(x,m) \right] \qquad (3.42)$$

$$f_1(x) = \max \left[\sum_{m=0}^{x} r(m)\, p(x,m) \right] \tag{3.43}$$

...re α, $0 < \alpha < 1$, is the discount factor.

For large n, the equation (3.42) can be written as

$$f(x) = \max \left[\sum_{m=0}^{x} r(m)\, p(x,m) + \alpha \sum_{m=0}^{x} f(x-m)\, p(x,m) \right] \tag{3.44}$$

...ce $0 < \alpha < 1$, we have

$$\left[1 - \alpha\, p(x,o) \right] f(x) = \max \left[\sum_{m=0}^{x} r(m)\, p(x,m) + \alpha \sum_{m=1}^{x} f(x-m) p(x,m) \right]$$

...that

$$f(x) = \max \left\{ \left[\sum_{m=0}^{x} r(m) p(x,m) + \alpha \sum_{m=1}^{x} f(x-m) p(x,m) \right] / \left[1 - \alpha\, p(x,o) \right] \right\} \tag{3.45}$$

...can observe that the assumption $p(x,o) \neq$ ... is not essential here.

We now consider the continuous version of the problem. Let $r(m)$ ... the value of the return when m units are invested during the period ...er consideration and let $F(x,m)$ denote the probability density ...ction of the amount of investment, m, during anyperiod when the decision-...ker has x units for investment. Let $f(x)$ be the maximum expected ...ounted return on investment when the decision maker has a capital of ...units and an optimal decision making plicy is employed. Then the ...responding recurrence relation is given by

$$f(x) = \max \left[\int_{0}^{x} r(m)\, dF(x,m) + \alpha \int_{0}^{x} f(x-m)\, dF(x,m) \right] \tag{3.46}$$

the analysis of the solution of (3.46) is an extension of earlier approach.

We have thus considered the problem of optimal decision rules for investment as a stochastic model (both discrete and continuous version) and the same is analysed with the help of functional equation approach of dynamic programming.