CHAPTER 4

DYNAMIC PROGRAMMING APPROACH TO INVENTORY TYPE PROBLEMS

SECTION I : THE WAREHOUSING PROBLEM

4.1.1 INTRODUCTION

One of the classical problems of linear programming is the so called warehousing problem which can be considered as an inventory type problem. The problem can be stated as follows : Given a warehouse with fixed capacity and an initial stock of a certain product which is subject to known seasonal price and cost variations, what is the optimal pattern of purchasing (or production), storage and sales ? The problem was initially formulated by Cahn [18] and Charnes and Cooper [20] gave an analytical approach to cover cases where more than one commodity is traded and nonlinear pricing is allowed. Bellman [10] and Dreyfus [35] have given the functional equation approach of dynamic programming for a single commodity model. Eastman [37] considered a multi-product model and gave the matrix formulation method to solve the problem. Also, he considered this problem as a special case of the shortest route problem. Here, we give a functional equation approach of dynamic programming for both single product and multi-product models. In addition to the commodity purchase costs and selling prices, we also consider the storage costs. These costs are assumed to be linear functions. To assist the computational aspect of the problem, a computer program is written in FORTRAN language. It is also run on a IBM 1620 computer.

6.1.2 FORMULATION OF THE MODEL – Single Product Case

Given the purchase costs and selling prices of a seasonal product in each of the n time periods, we want to find a schedule for purchases and sales so as to maximize the over-all profit. A warehouse of fixed capacity is available for storing the product between the time of purchase and sale. There is also a storage cost which is proportional to the units stored. We shall assume that the warehouse is initially empty and that the stock at the end of the period n should be zero.

Notations :

During the i^{th} period, let

s_i = unit selling price,

p_i = unit purchasing cost,

x_i = quantity purchased,

y_i = quantity sold,

h_i = unit storage (holding) cost.

The constraints are as follows :

(i) Buying constraint :- The stock on hand at the end of i^{th} period cannot exceed the warehouse capacity.

(ii) Selling constraint :- The quantity sold during the i^{th} period cannot exceed the amount available at the end of the period $(i-1)$.

(iii) Non-negativity constraint :- The quantity purchased or sold during any period are non-negative.

Let w be the warehouse capacity. Then, mathematically, the above constraints can be expressed as

$$\text{(i)} \quad \sum_{j=1}^{i} (x_j - y_j) \leq v \,, \qquad i = 1, \ldots, n$$

$$\text{(ii)} \quad y_i \leq \sum_{j=1}^{i-1} (x_j - y_j) \,, \qquad i = 1, \ldots, n \qquad (4.1)$$

$$\text{(iii)} \quad x_i \,, y_i \geq 0 \,, \qquad i = 1, \ldots, n$$

The problem is to determine the quantities x_i and y_i so as to maximize the profit function

$$P = \sum_{i=1}^{n} \left\{ (s_i y_i - p_i x_i) - h_i (x_i - y_i) \right\} \qquad (4.2)$$

subject to (4.1).

4.3 DYNAMIC PROGRAMMING APPROACH

Let u be the level of inventory attained at the end of the period under investigation. The choice of u will be called the " policy " for that period and we wish to determine the optimal sequence $u_1, u_2, \ldots, u_n$. Since the initial stock is zero, it is clear that the maximum profit is a function of the level of inventory at the end of the period under consideration and the duration of the process. Clearly, $u = x-y$ and it is less than or equal to the maximum capacity of the warehouse.

Let us now define

$$f_n(w) = \text{Max} \left\{ P \right\} \tag{4.3}$$

where the maximum is taken over all admissible values of x_i and y_i.

Using the principle of optimality of dynamic programming, the recurrence relation can be obtained as

$$f_n(w) = \underset{\substack{x_n, y_n \geq o \\ o \leq u \leq w}}{\text{Max}} \left\{ (s_n y_n - p_n x_n) - h_n(x_n - y_n) + f_{n-1}(u) \right\}$$

$$= \underset{o \leq u \leq w}{\text{Max}} \left\{ \underset{\substack{x_n, y_n \geq o \\ x_n - y_n = u}}{\text{Max}} \left[(s_n y_n - p_n x_n) - h_n(x_n - y_n) \right] + f_{n-1}(u) \right\}$$

$$= \underset{o \leq u \leq w}{\text{Max}} \left\{ R_n(u) + f_{n-1}(u) \right\} \tag{4.4}$$

where

$$R_n(u) = \underset{\substack{x_n, y_n \geq o \\ x_n - y_n = u}}{\text{Max}} \left\{ (s_n y_n - p_n x_n) - h_n(x_n - y_n) \right\} \tag{4.5}$$

Since $R_n(u)$, the profit during the n^{th} period, is a linear function in x_n and y_n, the maximum value of (4.5) can be found out in a simple way. Having found out $R_n(u)$, we can use (4.4) to find out the optimal purchasing and selling policy and the corresponding total profit over a planning horizon of n periods.

4.1.4 NUMERICAL EXAMPLE

Consider the following example, the data for which is given in

Table 4.1 where p_i and s_i are unit purchasing and selling prices. For simplicity, let us assume that $h_i = h = 1$ for all i , $i = 1, \ldots, 12$. Let w be the warehouse capacity and the initial stock be zero.

Table 4.1

i	1	2	3	4	5	6	7	8	9	10	11	12
p_i	18	17	15	16	17	21	22	18	19	18	17	18
s_i	14	15	17	18	20	22	19	17	18	20	22	21

Since the initial stock is zero, we have the following solution.

Period 1 - Since $s_2 < p_1$, we have

$x_1 = o$, $y_1 = o$ and $u_1 = o$. Hence $f_1(w) = o$.

Period 2 - Since $s_3 = p_2$ and $h = 1$, we have

$x_2 = o$, $y_2 = o$ and $u_2 = o$. Hence $f_2(w) = o$.

Period 3 - Since $s_4 > p_3$, we have

$x_3 = w$, $y_3 = o$ and $u_3 = w$. Hence $f_3(w) = -16\, w$.

Continuing in this way, we have

Period 4 - $x_4 = w$, $y_4 = w$ and $u_4 = w$. Hence $f_4(w) = -15w$.

Period 5 - $x_5 = w$, $y_5 = w$ and $u_5 = w$. Hence $f_5(w) = -13w$.

Period 6 - $x_6 = o$, $y_5 = w$ and $u_6 = o$. Hence $f_6(w) = 9\, w$.

Period 7 - $x_7 = o$, $y_7 = o$ and $u_7 = o$. Hence $f_7(w) = 9\, w$.

Period 8 - $x_8 = o$, $y_8 = o$ and $u_8 = o$. Hence $f_8(w) = 9\, w$.

Period 9 - $x_9 = w$, $y_9 = o$ and $u_9 = w$. Hence $f_9(w) = -11\, w$.

Period 10 - $x_{10} = w$, $y_{10} = w$ and $u_{10} = w$. Hence $f_{10}(w) = -10\ w$.

Period 11 - $x_{11} = w$, $y_{11} = w$ and $u_{11} = w$. Hence $f_{11}(w) = -6\ w$.

Period 12 - $x_{12} = o$, $y_{12} = w$ and $u_{12} = o$. Hence $f_{12}(w) = 15\ w$.

Hence the maximum profit over the entire planning horizon is equal to 15 w. The optimal policy can be stated as follows :

Buy the product during the periods 3, 4, 5, 9, 10 and 11 up to a maximum of w units and sell them during the following period. If $w = 100$ units, then the maximum profit will be Rs. 1500. The computer program (written in FORTRAN) is given in Appendex. The program has been run on a IBM 1620 computer system.

4.1.5 MULTIPRODUCT MODEL

Usually, the problem that is encountered is of designing a policy for a multiproduct model rather than for a single product model. The multiproduct model can be stated as follows :

Given purchase costs and selling prices for each of the m products in each of the n time periods, it is necessary to schedule a purchasing and selling policy so as to maximize the over-all profit. A warehouse of fixed capacity is available for storing the products between the time of purchase and the sale. It is assumed that the warehouse is initially empty and the stock at the end of the n^{th} period should be zero.

Notations :

Extending the notations used in the single product model, we write

s_i^k = unit selling price of k^{th} product during the i^{th} period, $k = 1, \ldots, m$; $i = 1, \ldots, n$.

p_i^k = unit purchasing price of k^{th} product during the i^{th} period, $k = 1, \ldots, m$; $i = 1, \ldots, n$.

x_i^k = quantity of the k^{th} product purchased during the i^{th} period, $k = 1, \ldots, m$; $i = 1, \ldots, n$.

y_i^k = quantity of the k^{th} product sold during the i^{th} period, $i = 1, \ldots, n$; $k = 1, \ldots, m$.

Let h_i be the unit holding (storage) cost during the i^{th} period which is assumed to be independent of the item stored and let w be the warehouse capacity.

The constraints, then, are

(i) Buying constraint :- The stock in hand of all the m products at the end of the i^{th} period can not exceed the warehouse capacity.

(ii) Selling constraint :- The amount of the k^{th} product sold during the i^{th} period can not exceed the amount of the commodity available at the end of the previous period.

(iii) Non-negativity constraints :- Quantities purchased or sold of any product during any period are non-negative.

The selling constraint stated above reflects the fact that the different commodities do not interact on the sales side.

The procedure developed for the single product model can be extended to encompass this problem. The profit function, to be maximized, can be written as

$$P = \sum_{i=1}^{n} \sum_{k=1}^{m} (s_i^k \; y_i^k - p_i^k \; x_i^k) - \sum_{i=1}^{n} h_i \sum_{k=1}^{m} (x_i^k - y_i^k) \qquad (4.6)$$

subject to

$$\text{(i)} \quad \sum_{j=1}^{i} \sum_{k=1}^{m} (x_j^k - y_j^k) \leq \omega , \quad i = 1, \ldots, n$$

$$\text{(ii)} \quad y_i^k \leq \sum_{j=1}^{i-1} (x_j^k - y_j^k) , \quad k = 1, \ldots, m ; \quad i = 1, \ldots, n \qquad (4.7)$$

$$\text{(iii)} \quad x_i^k , \; y_i^k \geq o , \qquad k = 1, \ldots, m ; \quad i = 1, \ldots, n.$$

4.1.6 DYNAMIC PROGRAMMING APPROACH

Let u be the combined level of inventory of m products at the end of the period under consideration, then

$$u = \sum_{k=1}^{m} (x_i^k - y_i^k) , \quad i = 1, \ldots, n \qquad (4.8)$$

and since the maximum capacity of the warehouse is w, $u \leq w$. Then using the functional equation approach of dynamic programming, we can write the recurrence relation for the n period profit as

$$f_n^m (w) = \max_{\substack{x_n^k, \, y_n^k \geq o \\ o \leq u \leq w}} \left\{ \sum_{k=1}^{m} (s_n^k \, y_n^k - p_n^k \, x_n^k) - h_n \sum_{k=1}^{m} (x_n^k - y_n^k) + f_{n-1}^m(u) \right\}$$

$$= \max_{0 \leq u \leq w} \left\{ \max_{\substack{x_n^k,\ y_n^k \geq 0 \\ \sum_{k=1}^{m} (x_n^k - y_n^k) = u}} \left[\sum_{k=1}^{m} (s_n^k y_n^k - p_n^k x_n^k) - h_n \sum_{k=1}^{m} (x_n^k - y_n^k) \right] + f_{n-1}^{m}(u) \right\} \qquad (4.9)$$

where $f_n^m(w)$ denotes the maximum profit over the n periods when there are m products and the capacity of the warehouse being equal to w .

We can write (4.9) as

$$f_n^m(w) = \max_{0 \leq u \leq w} \left\{ R_n^m(u) + f_{n-1}^m(u) \right\} \qquad (4.10)$$

where

$$R_n^m(u) = \max_{\substack{x_n^k,\ y_n^k \geq 0 \\ \sum_{k=1}^{m} (x_n^k - y_n^k) = u}} \left\{ \sum_{k=1}^{m} (s_n^k y_n^k - p_n^k x_n^k) - h_n \sum_{k=1}^{m} (x_n^k - y_n^k) \right\} \qquad (4.11)$$

$R_n^m(u)$ being the n^{th} period return when there are m products for consideration.

4.1.7 NUMERICAL EXAMPLE

Consider a six period, two-product problem where the purchasing prices and the selling prices are given (Table 4.2). The manufacturer (decision maker) can stock either of the two products or a combination of them up to the total capacity, w , of the warehouse. Given that the initial inventory of the two products is zero and it is desired that the final stock is also zero.

Table 4.2

Period i	Product 1		Product 2	
	p_i^1	s_i^1	p_i^2	s_i^2
1	28	23	37	22
2	27	37	33	43
3	25	30	26	26
4	31	21	32	37
5	37	42	39	49
6	33	38	32	46

Let $h_i = h = 1$ for all i. We shall use equations (4.11) and (4.10) to obtain the maximum profit for the six period problem subject to the constraints given in (4.7). The optimal decision policy is given in Table 4.3.

Table 4.3. Optimal purchasing and selling policy.

Period	Product 1		Product 2		Stock at the end of the period	Total profit
	Sell	Purchase	Sell	Purchase		
1	–	w	–	–	w	–29w
2	w	w	–	–	w	–20w
3	w	–	–	w	w	–17w
4	–	–	w	w	w	–13w
5	–	–	w	w	w	– 4w
6	–	–	w	–	–	42 w

Hence, the optimal policy can be stated as follows :

During period 1, purchase product 1 upto w units and do not purchase product 2. Since the initial inventory is zero, sell nothing. During period 2, sell all the products purchased during the previous period and purchase product 1 upto w units and do not purchase product 2. During period 3, sell all the products purchased during the previous period and purchase product 2 upto w units and do not purchase product 1. During 4th period, sell all the products purchased during the previous period and purchase product 2 upto w units and do not purchase product 1. During next period, sell all the products purchased during the previous period and purchase product 2 upto w units and do not purchase product 1. During the last period, sell all the products purchased in the previous period and purchase nothing.

SECTION II : MULTISTAGE INVENTORY PROBLEM

4.2.1 INTRODUCTION

An inventory can be defined as a stock of items which is held for the purpose of future production or sales by a manufacturing concern or by a dealer. Since inventories constitute an alternative to production or purchase during the future periods, the selection among the policies depend upon their relative profitabilities or losses. Some of the cost factors that determine the profit or loss are (i) the costs of ordering or manufacturing, (ii) holding or storage costs, (iii) unsatisfied demand or shortage penalty costs, (iv) revenues, (v) salvage costs, and (vi) discount rate.

The inventory model studied here is an extension and modification of the model formulated on the optimal inventory policy by Arrow, et.al. [3], Bellman, et.al. [16], Dvoretzky, et.al. [36] and Karlin [53].

4.2.2 FORMULATION OF THE PROBLEM

The problem that is considered here is concerned with the stocking of items to meet an uncertain demand, uncertain in the sense that only the probability with which the demand will be in any interval of real numbers is taken to be known, under the assumptions that there are various costs associated with over supply and under supply. At various specified times, there is an opportunity to order supply of a certain item, where the cost of ordering depends upon the size of the order and some other cost which is fixed per order. At various other times, demands are met from these stocks where the demand is not known in advance, but the probability distribution of demand is known. The following assumptions are made in the model :

(i) The known probability distribution function of demand is assumed to have a continuous positive density function $\phi(\xi)$, for $\xi > 0$.

(ii) Unless explicitly stated to the contrary, it is assumed that the time lag between the ordering and the delivery of a commodity is negligible.

(iii) The storage or handling cost, if any, is assumed to be a function of stock on hand at the end of the period, i.e., the storage cost is given by a function $h(y-\xi)$ if $y > \xi$ and is zero if $y < \xi$.

(iv) There is no backlog of excess of demand.

The cost of obtaining z units ordered at the beginning of the period during which the demand must be supplied from the stock, is assumed to have the form $k + c(z)$ where

$$c(z) = \begin{cases} c.z & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases}$$

where k is the ordering cost, c and k being constants.

If the amount of demand during the period, ξ, is greater than the amount in stock, y, then a positive penalty cost $p(\xi - y)$ is incurred. If $\xi \leq y$, then $p(\xi - y) = 0$.

When the stock is greater than the demand during a period, a positive storage cost $h(y-\xi)$ and a positive salvage gain $v(y-\xi)$ are incurred. For $\xi \geq y$, $h(y-\xi) = 0$ and $v(y-\xi) = 0$. The unit sale price of the items is assumed to be a constant, say r per unit.

4.2.3 DYNAMIC PROGRAMMING APPROACH

Let x denote the stock level at the beginning of the process. The problem is to determine an optimal ordering policy which will minimize the over-all loss, the process being carried over n periods. Let

$L_n(x)$ = the expected total loss for an n period process starting with an initial stock of x units and using an optimal ordering policy.

If we consider a single period process, then we get

$$L_1(x) = \underset{y \geq x}{\text{Min}} \Big\{ k + c(y-x) + \int_0^y \left[h(y-\xi) - v(y-\xi) - r\xi \right] \phi(\xi)\, d\xi + \int_y^\infty \left[p(\xi - y) - ry \right] \phi(\xi)\, d\xi \Big\} \qquad (4.12)$$

when a quantity $(y-x) \geq 0$ is ordered.

Using the principle of optimality, the recurrence relation for the n period process can be written as

$$L_n(x) = \underset{y \geq x}{\text{Min}} \Big\{ k + c(y-x) + \int_0^y \left[h(y-\xi) - v(y-\xi) - r\xi \right] \phi(\xi)\, d\xi + \int_y^\infty \left[p(\xi - y) - ry \right] \phi(\xi)\, d\xi + L_{n-1}(0) \int_y^\infty \phi(\xi)\, d\xi + \int_0^y L_{n-1}(y-\xi)\, \phi(\xi)\, d\xi \Big\} \qquad (4.13)$$

by taking into consideration the various cases corresponding to the possibility of an excess of demand over supply and an excess of stock over demand. For $n = 1$, (4.13) reduces to (4.12) with $f_o(x) = o$.

The general theory (Bellman [11]) implies that $L_n(x)$ converges to $L(x)$ and the optimal policy $y_n(x)$ converges to $y(x)$ as n becomes large. Let us introduce the concept of discounting the future costs. Let α, $o < \alpha < 1$, be a fixed discount factor for each period. Introducing α in (4.13) and as n becomes large, the functional equation reduces to the form

$$L(x) = \underset{y \geq x}{\text{Min}} \Big\{ k + c(y-x) + \int_o^y [h(y-\xi)-v(y-\xi)-r\xi]\phi(\xi)d\xi$$

$$+ \int_y^{\infty} [p(\xi-y)-ry]\phi(\xi)\,d\xi$$

$$+ \alpha [L(o)\int_y^{\infty}\phi(\xi)d\xi + \int_o^y L(y-\xi)\phi(\xi)d\xi] \Big\} \quad (4.14)$$

where $L(x)$ is the expected over-all discounted cost starting with an initial stock of x units and using an optimal ordering policy. If we assume that some of the items supplied upon demand may be partially returned, so that the demand of ξ items result in a return of $\rho\xi$ items, $o \leq \rho \leq 1$, which can be added to the stock, then the equation (4.14) reduces to the form

$$L(x) = \underset{y \geq x}{\text{Min}} \Big\{ k + c(y-x) + \int_o^y [h(y-\xi)-v(y-\xi)-r\xi]\phi(\xi)d\xi$$

$$+ \int_y^{\infty} [p(\xi-y) - ry]\phi(\xi)d\xi$$

$$+ \alpha [\int_y^{\infty} L(\rho\xi)\phi(\xi)d\xi + \int_o^y L(y-\xi+\rho\xi)\phi(\xi)d\xi] \Big\} \quad (4.15)$$

If we alter the assumption (iv) of section 4.2.2 and allow back-logging of excess of demand, the associated functional equation takes the form

$$L(x) = \min_{y \geq x} \Big\{ k + c(y-x) + \int_0^y \left[h(y-\xi) - v(y-\xi) - r\xi \right] \phi(\xi)\,d\xi$$

$$+ \int_y^\infty \left[p(\xi - y) - ry \right] \phi(\xi)\,d\xi + \alpha \int_0^\infty L(y-\xi)\,\phi(\xi)\,d\xi \Big\} \qquad (4.16)$$

4.2.4 SOLUTION OF THE PROBLEM

We shall now characterise the optimal ordering rule for the infinite stage problem. Rewriting equation (4.14), we have

$$L(x) = \min_{y \geq x} \Big\{ k + c(y-x) + \int_0^y \left[h(y-\xi) - v(y-\xi) - r\xi \right] \phi(\xi)\,d\xi$$

$$+ \int_y^\infty \left[p(\xi - y) - ry \right] \phi(\xi)\,d\xi + \alpha L(0) \int_0^\infty \phi(\xi)\,d\xi$$

$$+ \alpha \int_0^y L(y-\xi)\,\phi(\xi)\,d\xi \Big\} \qquad (4.17)$$

If the minimum is attained at $y > x$, then we have, under the assumptions that the cost functions are linear functions,

$$c + (h-v) \int_0^y \phi(\xi)\,d\xi - (p + r) \int_y^\infty \phi(\xi)\,d\xi$$

$$+ \alpha \int_0^y L'(y-\xi)\,\phi(\xi)\,d\xi = 0 \qquad (4.18)$$

Moreover, for this value of y, we have

$$L'(x) = -c \qquad (4.19)$$

Equation (4.18) can be simplified to obtain

$$(c + h - v) - (h-v+p+r)\int_y^{\infty}\phi(\xi)d\xi + \alpha\int_0^y L'(y-\xi)\phi(\xi)d\xi = o \quad (4.20)$$

Consider the equation

$$L(x) = \underset{y \geq x}{\text{Min}}\Big\{ k+c(y-x) + \int_0^y \big[h(y-\xi)-v(y-\xi)- r\xi\big]\phi(\xi)d\xi$$

$$+ \int_y^{\infty}\big[p(\xi-y) - ry\big]\phi(\xi)d\xi + \alpha L(o)\int_y^{\infty}\phi(\xi)d\xi$$

$$+ \alpha\int_0^y L(y-\xi)\,\phi(\xi)d\xi\Big\} \quad (4.21)$$

where we impose the conditions

(i) c,k,h,v, p and r are positive constants,

(ii) $\phi(\xi) > o$, $\int_0^{\infty}\phi(\xi)d\xi = 1$, $\int_0^{\infty}\xi\phi(\xi)d\xi < \infty$

(iii) $p + r > o$.

Let x^* be the unique root of

$$c+h-v = (h-v+p+r)\int_y^{\infty}\phi(\xi)d\xi + \alpha c\int_0^y\phi(\xi)d\xi$$

Then the optimal policy is given by

$$\begin{aligned} &(i) \quad \text{for } o \leq x < x^*, \quad y = x^* \\ &(ii) \quad \text{for } x \geq x^*, \quad y = x \end{aligned} \quad (4.22).$$

That is, the optimal stock level is x^*.

From equations (4.18) and (4.19), if we replace the term

$\int^{y} L'(y-\xi)\varphi(\xi)d\xi$ by $-e\int_{0}^{y}\phi(\xi)d\xi$, then the equation (4.20) reduces to

$$c+h-v-(h-v+p+r)\int_{y}^{\infty}\phi(\xi)d\xi - \alpha e\int_{0}^{y}\phi(\xi)d\xi = o \qquad (4.23)$$

Simplifying (4.23), we get

$$\int_{0}^{y}\varphi(\xi)d\xi = \frac{p+r-c}{h-v+p+r-\alpha c} \qquad (4.24)$$

which has only one root if $\phi(\xi) > o$. Let this root be x^{*}.

For $o \leq x \leq x^{*}$, we have

$$L(x) = k+c(x^{*}-x) + \int_{0}^{x^{*}}\left[h(x^{*}-\xi)-v(x^{*}-\xi) - r\xi\right]\phi(\xi)d\xi$$

$$+ \int_{x^{*}}^{\infty}\left[p(\xi-x^{*}) - r x^{*}\right]\phi(\xi)d\xi$$

$$+ \alpha\left[L(o)\int_{x^{*}}^{\infty}\phi(\xi)d\xi + \int_{0}^{x^{*}}L(x^{*}-\xi)\varphi(\xi)d\xi\right] \qquad (4.25)$$

and $L'(x) = -c$. Thus

$$L(x) = L(o) - cx \qquad (4.26)$$

Using (4.26) in (4.25) and substituting $x = o$ and simplifying, we get

$$L(o) = \Big\{k + cx^{*} + \int_{0}^{x^{*}}\left[h(x^{*}-\xi) - v(x^{*}-\xi) - r\xi\right]\varphi(\xi)d\xi$$

$$+\int_{x^{*}}^{\infty}\left[p(\xi-x^{*})-rx^{*}\right]\phi(\xi)d\xi - \alpha c\int_{0}^{x^{*}}(x^{*}-\xi)\phi(\xi)d\xi\Big\}/(1-\alpha) \qquad (4.27)$$

Knowing L(o) from (4.27), we can obtain the value of L(x) from (4.26). For $x > x^*$, we have

$$L(x) = \int_0^x [h(x-\xi)-v(x-\xi)-r\xi]\phi(\xi)d\xi + \int_x^\infty [p(\xi-x)-rx]\phi(\xi)d\xi$$

$$+\alpha [L(o)\int_x^\infty \phi(\xi)d\xi + \int_0^x L(x-\xi)\phi(\xi)d\xi] \qquad (4.28)$$

Since L(o) is known, we can rewrite (4.28) as

$$L(x) = Z(x) + \alpha\int_0^x L(x-\xi)\phi(\xi)d\xi \qquad (4.29)$$

where $Z(x)$ is a function in x. Equation (4.29) may be written as

$$L(x) = Z(x) + \alpha\int_0^{x-x^*} L(x-\xi)\phi(\xi)d\xi + \alpha\int_{x-x^*}^{x} L(x-\xi)\phi(\xi)d\xi$$

In the interval $(x-x^*, x)$, $L(x-\xi)$ is known, hence, writing $\alpha\int_{x-x^*}^{x}$ $L(x-\xi)\phi(\xi)d\xi$ and $Z(x)$ together as $l(x)$, we have

$$L(x) = l(x) + \alpha\int_0^{x-x^*} L(x-\xi)\phi(\xi)d\xi \qquad (4.30)$$

Replacing $x-x^*$ by z and $L(x^*+z)$ by $g(z)$, we get

$$g(z) = l(x^*+z) + \alpha\int_0^z g(z-\xi)\phi(\xi)d\xi \qquad (4.31)$$

for $z \geq o$, which is a simple renewal equation.

Theorem (Feller [39]) : Suppose that $f(t)$ is a density function and that $a(x)$ is continuous. Then, the integral equation

$$u(x) = a(x) + \int_0^x u(x-t)f(t)dt, \quad (x \geq o)$$

possesses a unique solution, and if $a(x) \geq o$ for all $x \geq o$, then $f(x) \geq o$.

Using the above theorem to equation (4.31), we can deduce that $g(s)$ exists and that $g(s) \geq o$ since $l(x^* + s) \geq o$.

4.2.5 EXAMPLE

We shall apply the above approach to the following problem :

Consider that the demand for milk at a milk-booth is of the negative exponential with mean 50 cans. The cost functions are assumed to be as follows :

$$k + c(y - x) = 2.5 + 10.8(y-x)$$

$$h(y - \xi) = 1.63(y - \xi)$$

$$v(y - \xi) = 0.48(y - \xi)$$

$$p(\xi - y) = 3.04(\xi - y)$$

$$r = 12.5$$

$$\alpha = 0.95$$

The problem is to find the optimal stock level. We are given

$$\phi(\xi)d\xi = \frac{1}{50} e^{-\xi/50} d\xi$$

Substituting the above values in relation (4.24), we have

$$\frac{1}{50} \int_o^{x^*} e^{-\xi/50} d\xi = 0.73717$$

Thus, we get

$$e^{-x^*/50} = 0.26283$$

i.e., $x^* = 67.0$ (approximately.)

Hence the optimal stock level is 67 cans.