Chapter-II*

ON SOME CONTINUED FRACTION IDENTITIES OF SRINIVASA RAMANUJAN

1. Introduction.

The following continued fraction identities (1)-(3) and (I) found in the "lost" notebook of Ramanujan (terminology due to G.E. Andrews [5]), contain as special cases many of his other identities:

$$(1)\quad \frac{G(0,\lambda q,b,q)}{G(0,\lambda,b,q)} = \frac{1}{1+}\,\frac{\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\cdots\frac{\lambda q^{2n+1}}{1+}\,\frac{bq^{n+1}+\lambda q^{2n+2}}{1+}\cdots$$

$$(2)\qquad = \frac{1}{1+}\,\frac{\lambda q}{1+bq+}\,\frac{\lambda q^2}{1+bq^2+\cdots}\,\frac{\lambda q^n}{1+bq^n+\cdots}$$

$$(3)\qquad = \frac{1}{1-b+}\,\frac{b+\lambda q}{1-b+\cdots}\,\frac{b+\lambda q^n}{1-b+}\cdots$$

and, more generally

$$(\mathrm{I})\quad \frac{G(aq,\lambda q,b,q)}{G(a,\lambda,b,q)}$$

$$= \frac{1}{1+}\,\frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\cdots\frac{aq^{n+1}+\lambda q^{2n+1}}{1+}\,\frac{bq^{n+1}+\lambda q^{2n+2}}{1+}\cdots$$

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* Reference [12] is based on this chapter.

where

$$(4)\qquad G(a,\lambda,b,q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} \left(-\frac{\lambda}{a}\right)_n a^n}{(q)_n(-bq)_n} .$$

It is easily seen that (1)-(3) are themselves special cases respectively of (I) above and (II) and (III) below:

$$(\mathrm{II})\qquad \frac{G(aq,\lambda q,b,q)}{G(a,\lambda,b,q)} = \frac{1}{1+}\;\frac{aq+\lambda q}{1-aq+bq+}\cdots\;\frac{aq+\lambda q^n}{1-aq+bq^n+}\cdots$$

$$(\mathrm{III})\qquad = \frac{1}{1-b+aq+}\;\frac{b+\lambda q}{1-b+aq^2+}\cdots\;\frac{b+\lambda q^n}{1-b+aq^{n+1}+}\cdots$$

(III) is an identity due to Hirschhorn [25]. Identity (I), and hence (1), has been proved independently by Andrews [5] and Hirschhorn [26]. Andrews has employed G and some auxiliary functions and a transformation of E. Heine; and Hirschhorn has proved it by obtaining a closed form for the n-th convergent. While Andrews [7] has given a separate proof (which is essentially the same as Ramamani's unpublished proof [30]) of the "slightly tricky" identity (2) he has extracted (3) as a particular case of (III) which Hirschhorn [25] has proved by finding a closed form for the n-th convergent. Many other identities of Kamanujan also follow as pointed out by Andrews and Hirschhorn. However, we have not come across (II) nor the proofs of the following Ramanujan identities (IV) and (5) found in the "lost" notebook [35]:

$$\text{(IV)}\quad \frac{G(aq,\lambda q,b,q)}{G(a,\lambda,b,q)} = \frac{1}{1+aq+}\;\frac{\lambda q-abq^{2}}{1+q(aq+b)+\ldots}\;\frac{\lambda q^{n}-abq^{2n}}{1+q^{n}(aq+b)+\ldots},$$

$$(5)\quad \frac{1}{a+c-}\;\frac{ab}{a+b+cq-}\;\frac{ab}{a+b+cq^{2}-\ldots}\;\frac{ab}{a+b+cq^{n}-\ldots}$$

$$= \frac{1}{c-b+a+}\;\frac{bc}{c-b+\frac{a}{q}+}\;\frac{bc}{c-b+\frac{a}{4^{2}}+\ldots}\;\frac{bc}{c-b+\frac{a}{q^{n}}+\ldots}$$

In what follows we employ as auxiliary function instead of $G(a,\lambda,b,q)$ a multiple of it, namely,

$$(4^{*})\quad g(a,\lambda,b,q) = (-bq)_{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}\left(-\frac{\lambda}{a}\right)_{n}a^{n}}{(q)_{n}\,(-bq)_{n}}$$

and will thereby be able to give a simple, unified and self-contained approach to proving (I), (II), (III), (IV) and (5). We may observe that in all the identities (I)-(IV) and (1)-(3) we may replace G by g. We deduce (I)-(IV) directly (Section 3) from three easily proved canonical functional relations (6)-(8) for g (Section 2) and extract (5) as a particular case of the identity (II)= (IV) with $\lambda=0$.

We also obtain (Section 4) two other consequences of the functional relations for g, namely, the Ramanujan's basic hypergeometric transformation:

$$(-bq)_\infty \sum_{n=0}^{\infty} \frac{\lambda^n q^{n^2}}{(q)_n(-bq)_n} = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-\frac{\lambda}{b})_n b^n}{(q)_n}$$

and more generally,

$$g(a,\lambda,b,q) = g(b,\lambda,a,q).$$

We further indicate (Section 5) that by using the former identity, some special y-continued fraction identities of Ramanujan follow from (1).

2. Canonical functional relations satisfied by g

Lemma 1 (Key Lemma). If $|q| < 1$, then g satisfies the following functional relations:

$$g(a,\lambda,b,q) - g(aq,\lambda,b,q) = aq\, g(aq,\lambda q,bq,q) \tag{6}$$

$$g(a,\lambda,b,q) - g(a,\lambda q,b,q) = \lambda q\, g(aq,\lambda q^2,bq,q) \tag{7}$$

$$g(a,\lambda,b,q) - g(a,\lambda,bq,q) = bq\, g(aq,\lambda q,bq,q)\ . \tag{8}$$

Proof: Since $(-\frac{\lambda}{a})_n - q^n(-\frac{\lambda}{aq})_n$ equals 0 if n is 0 and $(-\frac{\lambda}{a})_{n-1}(1-q^n)$ if $n \geq 1$ as can be easily verified, we have

left side of (6)

$$= (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}\, a^n}{(q)_n(-bq)_n} \cdot \left[(-\frac{\lambda}{a})_n - q^n(-\frac{\lambda}{aq})_n\right]$$

$$= aq(-bq^2)_\infty \sum_{n=1}^{\infty} \frac{q^{\frac{n(n-1)}{2}}(aq)^{n-1}(-\frac{\lambda}{a})_{n-1}}{(q)_{n-1}(-bq^2)_{n-1}}$$

$= aq\, g(aq,\lambda q,bq,q)$, proving (6).

In the penultimate step we have used the obvious identity

$$\frac{(-bq)_\infty}{(-bq)_n} = \frac{(-bq^2)_\infty}{(-bq^2)_{n-1}}\ .$$

Relations (7) and (8) follow in exactly the same way on using $(-\frac{\lambda}{a})_n - (-\frac{\lambda q}{a})_n$ equals o if n is o and $\frac{\lambda}{a}(-\frac{\lambda q}{a})_{n-1}(1-q^n)$ if $n \geqq 1$, and

$$\frac{(-bq)_\infty}{(-bq)_n} - \frac{(-bq^2)_\infty}{(-bq^2)_n} \qquad \frac{(-bq^2)_\infty}{(-bq^2)_n}\, bq^{n+1}.$$

Lemmas 2-5 proved below are simple combinations of relations (6)-(8). Also, Theorems 1-4 follow directly from Lemmas 2-5 respectively in a simple manner.

3. Continued fraction developments I-IV and (5)—their proofs

Lemma 2. g satisfies

$$(9)\quad g(a,\lambda,b,q) = g(aq,\lambda q,b,q) + (aq+\lambda q)\, g(aq,\lambda q^2,bq,q)$$

$$(10)\quad g(a,\lambda,b,q) = g(a,\lambda q,bq,q) + (bq+\lambda q)\, g(aq,\lambda q^2,bq,q).$$

Proof: Changing λ to λq in (6) and adding to (7) gives (9), while changing λ to λq in (8) and adding to (7) gives (10).

Theorem 1 If $|q|<1$, then

$$(\mathrm{I})\quad \rho = \frac{1}{1+}\;\frac{aq+\lambda q}{1+}\;\frac{bq+\lambda q^2}{1+}\;\cdots\;\frac{aq^{n+1}+\lambda q^{2n+1}}{1+}\;\frac{bq^{n+1}+\lambda q^{2n+2}}{1+}\;\cdots$$

where

$$\rho = \frac{G(aq,\lambda q,b,q)}{G(a,\lambda,b,q)} = \frac{g(aq,\lambda q,b,q)}{g(a,\lambda,b,q)}.$$

Proof: Changing a to aq^n, λ to λq^{2n}, b to bq^n in (9) and changing a to aq^{n+1}, λ to λq^{2n+1} and b to bq^n in (10) we can write (9)'and (10) respectively as:

$$Q_n \equiv \frac{g(aq^n,\lambda q^{2n},bq^n,q)}{g(aq^{n+1},\lambda q^{2n+1},bq^n,q)} = 1+\frac{aq^{n+1}+\lambda q^{2n+1}}{Q'_n}$$

$$Q'_n \equiv \frac{g(aq^{n+1},\lambda q^{2n+1},bq^n,q)}{g(aq^{n+1},\lambda q^{2n+2},bq^{n+1},q)} = 1+\frac{bq^{n+1}+\lambda q^{2n+2}}{Q_{n+1}}.$$

Iterating the last two identities alternately with n=0,1,2...

we have (I). Convergence of the continued fraction follows Since Q_n, $Q'_n \longrightarrow 1$ as $n \longrightarrow \infty$.

Lemma 3. g satisfies

$$(11)\quad g(a,\lambda,b,q) = g(aq,\lambda q,b,q) + (aq+\lambda q)\, g(aq,\lambda q^2,bq,q)$$

$$(12)\quad g(aq,\lambda,b,q) = (1-aq+bq)\, g(aq,\lambda q,bq,q) + (aq+\lambda q)\, g(aq,\lambda q^2,bq^2,q).$$

Proof: Changing λ to λq in (6) and adding it to (7) we have (11). Changing λ to λq and b to bq in (6), b to bq in (7), taking the negative of (6) and adding these three equations to (8) we deduce (12).

Theorem 2. If $|q| < 1$ then,

$$(\mathrm{II})\quad \xi = \frac{1}{1+}\;\frac{aq+\lambda q}{1-aq+bq+}\cdots\;\frac{aq+\lambda q^n}{1-aq+bq^n+}\cdots$$

where ξ is as in 'Theorem 1.

Proof: (11) can be written as

$$(13)\quad \frac{g(aq,\lambda q,b,q)}{g(a,\lambda,b,q)} = \frac{1}{1+}\;\frac{aq+\lambda q}{\dfrac{g(aq,\lambda q,b,q)}{g(aq,\lambda q^2,bq,q)}}$$

Changing λ to λq^{n+1} and b to bq^n, we can write (12) as

$$S_n \equiv \frac{g(aq,\lambda q^{n+1},\ bq^n,q)}{g(aq,\lambda q^{n+2},bq^{n+1},q)} = (1-aq+bq^{n+1}) + \frac{aq+\lambda q^{n+2}}{S_{n+1}} .$$

Iterating thid with n= 0,1,2 . . . and using (13) we have (II). Convergence of (II) follows since $S_n \longrightarrow 1$ as $n \longrightarrow$ co when $|q|<1$.

Lemma 4. g satisfies

(14) $$g(a,\lambda,bq,q) = (1-bq+aq)\ g(aq,\lambda q,bq,q) + (bq+\lambda q)\ g(aq^2,\lambda q^2,bq,q)$$

Proof: Changing a to aq in (7), a to aq and λ to λq in (8), taking the negative of (8) and adding these three equations to (6) we deduce (14).

Theorem 3. If $|q|<1$ then,

(III) $$f = \begin{matrix} = - A - - \\ 1-b+aq+ \end{matrix} \quad \frac{b+\lambda q}{1-b+aq^2+\dots} \quad \frac{b+\lambda q^n}{1-b+aq^{n+1}+\dots}$$

where ρ is as in Theorem 1.

Proof: Changing a to aq^n, λ to λq^n and b to $\frac{b}{q}$ we can write (14) as

$$T_n \equiv \frac{g(aq^n,\lambda q^n,b,q)}{g(aq^{n+1},\lambda q^{n+1},b,q)} = (1-b+aq^{n+1}) + \frac{b+\lambda q^{n+1}}{T_{n+1}}$$

Iterating this with n = 0,1,2 . . . we have (III). Convergence of (III) follows as in the proof of Theorem 2.

Lemma 5. g satisfies

$$(15)\quad g(a,\lambda,bq,q) = (1+aq)\, g(aq,\lambda q,bq,q) + (\lambda q-abq^3)\, g(aq^2,\lambda q^2,bq^2,q)$$

$$(16)\quad g(aq,\lambda q,b,q) = \left\{1+q(aq+b)\right\} g(aq^2,\lambda q^2,bq,q) + (\lambda q^2-abq^4)\, g(aq^3,\lambda q^3,bq^2,q).$$

Proof: Change a to aq, λ to λq, b to bq in (6) and multiply the- result by $-bq$; change a to aq, b to bq in (7); change a to aq in (8); take the negative of (8) and add all these to (6) to obtain (15). Change a to aq, λ to λq in (15); change a to aq and λ to λq in (8) and add to get (16).

Theorem 4. If $|q|<1$ then,

$$(\mathrm{IV})\quad \rho = \frac{1}{1+aq+}\;\frac{\lambda q-abq^2}{1+q(aq+b)+}\cdots\frac{\lambda q^n-abq^{2n}}{1+q^n(aq+b)+}\cdots$$

where ρ is as in 'Theorem 1.

Proof: Changing b to $\frac{b}{q}$, we can write (15) as

$$(17)\quad \frac{g(aq,\lambda q,b,q)}{g(a,\lambda,b,q)} = \frac{1}{1+aq+\dfrac{\lambda q-abq^2}{\dfrac{g(aq,\lambda q,b,q)}{g(aq^2,\lambda q^2,bq,q)}}}$$

Changing a, λ, b to aq^{n-1}, λq^{n-1} and by^{n-1} respectively, we can write (16) as

$$(18)\quad U_n \equiv \frac{g(aq^n, \lambda q^n, bq^{n-1}, q)}{g(aq^{n+1}, \lambda q^{n+1}, bq^n, q)} = 1 + q^n(aq+b) + \frac{\lambda q^{n+1} - abq^{2n+2}}{U_{n+1}}$$

Iterating (18) with $n = 1, 2 \ldots$ and using (17) we have (IV). Convergence of (IV) follows since $U_n \longrightarrow 1$ as $n \longrightarrow \infty$ when $|q| < 1$.

Theorem 5. If $|q| < 1$, then

$$(5)\quad \frac{1}{a+c-}\;\frac{ab}{a+b+cq-}\cdots\;\frac{ah}{a+b+cq^n-}\cdots$$

$$= \frac{1}{c-b+a+}\;\frac{bc}{c-b+\frac{a}{q}+}\cdots\;\frac{bc}{c-b+\frac{a}{q^n}+}\cdots$$

Proof: Changing $\lambda = 0$, a to $-\frac{b}{aq}$ and b to $\frac{c}{a}$ in (II)= (IV) and taking reciprocal we have

$$\frac{g(-\frac{b}{aq}, 0, \frac{c}{a}, q)}{g(-\frac{b}{a}, 0, \frac{c}{a}, q)} = 1 + \frac{-b/a}{1+\frac{b+cq}{a}+}\cdots\;\frac{-b/a}{1+\frac{b+cq^n}{a}+}\cdots$$

$$= \left(1 - \frac{b}{a}\right) + \frac{bcq/a^2}{1+\frac{q(c-b)}{a}+}\cdots\;\frac{bcq^{2n-1}/a2}{1+\frac{q^n(c-b)}{a}+}\cdots,$$

Multiplying this set of equations by a throughout and adding c throughout we have

$$c+\frac{a\,g(-\frac{b}{aq},o,\frac{c}{a},q)}{g(-\frac{b}{a},o,\frac{c}{a},q)}=(a+c)+\frac{-ab}{a+b+cq+\ldots}\quad\frac{-ab}{a+b+cq^{n}+\ldots}$$

$$=(a+c-b)+\frac{bcq}{a+(c-b)q+\ldots}\;\frac{bcq^{2n-1}}{a+(c-b)q^{n}+\ldots}$$

We complete the proof by taking the reciprocal through the last equations. In addition to proving (5) we have thus obtained that each side of (5) equals

$$\left\{c+\frac{a\,g(-\frac{b}{aq},o,\frac{c}{a},q)}{g(-\frac{b}{a},o,\frac{c}{a},q)}\right\}^{-1}.$$

4. A basic hypergeometric transformation of Ramanujan and a generalization

The transformation

$$(19)\quad (-bq)_\infty \sum_{n=o}^{\infty} \frac{q^{n^2}\lambda^n}{(q)_n(-bq)_n} = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}\left(-\frac{\lambda}{b}\right)_n b^n}{(q)_n}$$

or, what is the same,

$$(19)'\quad g(o,\lambda,b,q) = g(b,\lambda,o,q)$$

with g as in (4*), is found in Ramanujan's second notebook [34, Vol.II, p.194, Entry 9]. A particular case (b=1, $\lambda=-a$) of (19) was first posed as an Advanced problem by Carlitz [13] who also proved the general case [13] by employing Euler's expansion for $(a)_r$ as a ploynomial in a. Ramamani [30] has given a proof of (19) by obtaining two functional relations for the right hand side. Andrews [5] has shown that (13) is a limiting case of an identity of Rogers. Ramamani and Venkatachaliengar [31] have observed that (19) can be obtained as a limiting case of an identity of Heine.

In this section we make use of the functional relations (6)-(8) satisfied by

$$(4^*)\ g(a,\lambda,b,q) = (-bq)_\infty \sum_{n=o}^{\infty} \frac{q^{\frac{n(n+1)}{2}}\left(-\frac{\lambda}{a}\right)_n a^n}{(q)_n(-bq)_n}$$

not only to prove (19) butalso to show that (19) implies a more general identity

$$(20)\quad (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}\left(-\frac{\lambda}{a}\right)_n a^n}{(q)_n(-bq)_n} = (-aq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}\left(-\frac{\lambda}{b}\right)_n b^n}{(q)_n(-aq)_n}$$

or what is the same

$$(20)' \quad g(a,\lambda,b,q) = g(b,\lambda,a,q).$$

Theorem 6. Identity (19) holds.

Proof: Setting

$$(21)\quad g(o,bt,b,q) = \sum_{n=o}^{\infty} \beta_n(t,q)b^n$$

and putting $a=o$, $\lambda = bt$ in (12) and comparing coefficients we have

$$(1-q^n)\beta_n = q^n(1+tq^{n-1})\beta_{n-1},\ n = 1,2\ .\ .\ .\ .$$

Iterating this and noting that $\beta_o = g(o,o,o,q) = 1$ we get

$$\beta_n = \frac{q^{\frac{n(n+1)}{2}}(-t)_n}{(q)_n} = \frac{q^{\frac{n(n+1)}{2}}\left(-\frac{\lambda}{b}\right)_n}{(q)_n}.$$

Substituting this in (21) we get (19).

Theorem 7. Identity (20) holds.

Proof: Making use of the functional relations (6)-(8) we show that (19) implies (20). Substituting

$$g(a,\lambda,b,q) = \sum_{n=0}^{\infty} \beta_n(a,\lambda)b^n$$

and

$$g(a,\lambda,b,q) = \sum_{n=0}^{\infty} \alpha_n(\lambda,b)a^n$$

in (3) and (6) respectively and comparing coefficients we have

$$\beta_n(a,\lambda) = \frac{q^n}{1-q^n}\beta_{n-1}(aq,\lambda q)$$

and

$$\alpha_n(\lambda,b) = \frac{q^n}{1-q^n}\alpha_{n-1}(\lambda q,bq).$$

Iterating this we have

$$\beta_n(a,\lambda) = \frac{q^{\frac{n(n+1)}{2}}}{(q)_n}\beta_0(aq^n,\lambda q^n) = \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} g(aq^n,\lambda q^n,0,q)$$

and

$$\alpha_n(\lambda,b) = \frac{q^{\frac{n(n+1)}{2}}}{(q)_n}\alpha_0(\lambda q^n,bq^n) = \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} g(0,\lambda q^n,bq^n,q).$$

These imply

$$(22)\quad g(a,\lambda,b,q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q)_n}\, g(aq^n, \lambda q^n, 0, q)\, b^n$$

and

$$(23)\quad g(a,\lambda,b,q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q)_n}\, g(0, \lambda q^n, bq^n, q)\, a^n .$$

Interchanging a and b and using (19)', (23) becomes

$$g(b,\lambda,a,q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q)_n}\, g(aq^n, \lambda q^n, 0, q)\, b^n .$$

This with (22) proves (20).

A simple alternate proof of (20) but dependent on the well-known Euler's formulae for the expansions of $(a+\lambda)(a+\lambda q)\ldots(a+\lambda q^{n-1})$ and of $(-bq^{n+1})_\infty$ in powers of λ and b respectively, can be given.

5. Some corollaries:

We now give some corollaries which are deducible from the results of earlier sections. The identities

(24) $$(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n^2} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}}$$

and

(25) $$(q,q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n}} = \sum_{n=0}^{\infty} (-1)^n q^{n^2}$$

are stated by Ramanujan as corollaries to (19). In fact, we obtain (24) by putting $b= -1$ and $\lambda = q$ in (19). For (25) change q to q^2 and then put $b = -\frac{1}{q}$ and $\lambda = q$ in (19).

As noted by Andrews [5] the identity,

(26) $$\sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \lambda^n$$

$$= \frac{1}{1+}\ \frac{\lambda q}{1+}\ \frac{\lambda(q^2-q)}{1+} \ \cdots\ \frac{\lambda q^{2n+1}}{1-t}\ \frac{\lambda(q^{2n+2}-q^{n+1})}{1+}\ \cdots$$

listed by Ramanujan in his notebook [34, Vol.II, p.195, Entry 13], is a special case of (1) with $b= -\lambda$ provided we use (19). But, unlike us, he obtains (19) as a limiting case of an identity of Rogers. The special case of (26) with $\lambda=1$ is an identity of Eisenstein [16]. The Euler's identity

$$(27)\qquad (-b)_{\infty} = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}\, b^{n}}{(q)_{n}}$$

is itself a special case of (19), for, it can be written in the form $g(o,o,\frac{b}{q},q) = g(\frac{b}{q},o,o,q)$. As observed by Andrews [5] the identity

$$\prod_{n=0}^{\infty} \frac{(1-q^{2n+1})}{(1-q^{4n+2})^{2}} = \frac{1}{1+}\;\frac{q}{1+}\;\frac{q+q^{2}}{1+}\;\cdots\;\frac{q^{2n+1}}{1+}\;\frac{q^{n+1}+q^{2n+2}}{1+}\;\cdots$$

found in Ramanujan's 'lost' notebook [35] is also a special case of (1) with $b=1, \lambda=1$ provided we use (27). Andrews has also obtained a few other corollaries [5, pp.105-106] of (1) employing the works of Slater [39,40] and Rogers [37].