Chapter III*

A STUDY OF SOME BASIC HYPERGEOMETRIC TRANSFORMATIONS STATED BY RAMANUJAN LEADING TO GENERALIZATIONS OF SOME CONTINUED FRACTIONS CONSIDERED IN CHAPTER II

1. Introduction;

In Chapter II we gave a proof of the transformation

$$(1)\quad (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-\frac{\lambda}{a})_n a^n}{(q)_n (-bq)_n}$$

$$= (-aq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-\frac{\lambda}{b})_n b^n}{(q)_n (-aq)_n}$$

or, what is the same,

$$(2)\quad g(a,\lambda,b,q) = g(b,\lambda,a,q)$$

where

$$(3)\quad g(a,\lambda,b,q) = (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-\frac{\lambda}{a})_n a^n}{(q)_n (-bq)_n},$$

as a consequence of three canonical functional relations (6)-(8) of that Chapter satisfied by g. The present Chapter consists mainly of two related topics. In the first part

* Reference [12-A] is based on this Chapter.

(Section 2) we obtain two generalizations of (2), namely,

$$(4) \quad \tilde{g}(a, \lambda, b, c, q) = \tilde{g}(b, \lambda, a, c, q)$$

where

$$(5) \quad \tilde{g}(a, \lambda, b, c, q) = (-bq)_\infty \left(-\frac{aq}{c}\right)_\infty \sum_{n=0}^{\infty} \frac{\left(-\frac{\lambda}{a}\right)_n (c)_n \left(-\frac{aq}{c}\right)^n}{(q)_n (-bq)_n}$$

and

$$(6) \quad \hat{g}(a, \lambda, b, d, q) = \hat{g}(b, \lambda, a, d, q)$$

where

$$(7) \quad \hat{g}(a, \lambda, b, d, q) = (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} \left(-\frac{\lambda}{a}\right)_n \left(-\frac{db}{\lambda}\right)_n a^n}{(q)_n (-bq)_n (d)_n}$$

To obtain (4) and (6) we use two other identities found in Ramanujan's works [34, Vol. II, pp. 103-104, Entries 2 and 6] namely,

$$(8) \quad \frac{(-b)_\infty}{(a)_\infty} = \sum_{n=0}^{\infty} \frac{\left(-\frac{b}{a}\right)_n a^n}{(q)_n}$$

and

$$(9) \quad \frac{(d)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n \left(\frac{b}{a}\right)_n a^n}{(q)_n (d)_n} = \frac{(b)_\infty}{(a)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n \left(\frac{d}{c}\right)_n c^n}{(q)_n (b)_n} .$$

Identity (8) is indeed equivalent to the q-binomial theorem of Euler [17] and (9) is equivalent to an identity due to Heine [23]. Proofs of (t-j) and (9) can be found in standard texts [4, pp.19-20], and in Appendix-A we sketch a proof of (8) and one of (9) based on (8). In the course of our proof of (6) we obtain another identity of Ramanujan [34, Vol.II, p.194, Entry 8], namely,

$$(10)\qquad \frac{(a)_\infty}{(b)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n (\frac{b}{a})_n a^n}{(q)_n (d)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} (\frac{b}{a})_n (\frac{d}{c})_n (ac)^n}{(q)_n (b)_n (d)_n} .$$

In the second part of this chapter (Sections 3 and 4) we ostain in a simple manner four functional relations for $\tilde{g}(a,\lambda,b,c,q)$ and thereby generalizations of the continued fraction identities (I) and (II) of Chapter II. Tnese generalized identities are

$$(11)\qquad \frac{\tilde{g}(aq,\lambda q,b,c,q)}{\tilde{g}(a,\lambda,b,c,q)}$$

$$= \frac{1}{1+\frac{aq}{c}+} \; \frac{(aq+\lambda q)(1-\frac{1}{c})}{1+} \; \frac{bq+\lambda q^2}{1+\frac{aq}{c}+} \cdots$$

$$\frac{(aq^{n+1}+\lambda q^{2n+1})(1-\frac{1}{cq^n}) \quad bq^{n+1}+\lambda q^{2n+2}}{1+ \qquad\qquad 1+\frac{aq}{c}+} \cdots$$

and

$$(12)\qquad = \frac{1}{1+\frac{aq}{c}+} \; \frac{N_1}{D_1+} \; \frac{N_2}{D_2+} \cdots \frac{N_n}{D_n+} \cdots$$

where

$$(13) \qquad N_1 = (aq+\lambda q)\left(1-\frac{1}{c}\right),$$

$$D_1 + \frac{(1-aq+bq)+\frac{1}{c}(a+aq+\lambda q)}{\left(1+\frac{aq}{c}\right)}$$

and; for $n = 2, 3, \ldots$

$$N_n = \frac{(aq+\lambda q^n)\left(1-\frac{1}{cq^{n-1}}\right)}{1+\frac{aq}{cq^{n-2}}}$$

$$D_n = \frac{(1-aq+bq^n)+\frac{1}{cq^{n-1}}\,(a+aq+\lambda q^n)}{1+\frac{aq}{cq^{n-1}}}.$$

In the rest of this Chapter (Section 5) we give an elementary proof of another continued fraction identity of Ramanujan [34, Vol.II, p.195, Entry 11] related to (8):

$$(14) \qquad \frac{(-a)_\infty(b)_\infty-(a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty+(a)_\infty(-b)_\infty}$$

$$= \frac{a-b}{1-q+}\;\frac{(a-bq)(aq-b)}{1-q^3+}\;\frac{q(a-bq^2)(aq^2-b)}{1-q^5+}\;\cdots$$

$$\frac{q^{n-1}(a-bq^n)(aq^n-b)}{1-q^{2n+1}+}\;\cdots.$$

2. <u>'Two generalizations of tne transformation $g(a,\lambda,b,q) = g(b,\lambda,a,q)$</u>

<u>Theorem 1</u>. If $|q|<1$, $\left|\frac{aq}{c}\right|<1$ and $\left|\frac{bq}{c}\right|<1$ then identity (4) holds.

<u>Proof:</u> Changing c to B, b to At, d to C and a to t in (9) we have the Heine's transformation, namely,

$$(15) \quad \sum_{n=0}^{\infty} \frac{(A)_n(B)_n}{(q)_n(C)_n} t^n = \frac{(B)_\infty (At)_\infty}{(C)_\infty (t)} \sum_{m=0}^{\infty} \frac{(\frac{C}{B})_m (t)_m B^m}{(q)_m (At)_m}$$

when $|q|$, $|t|$ and $|B|<1$.

Since the left side of (15) is symmetric in A and 3, so must be the right hand side and hence

$$(At)_\infty (B)_\infty \sum_{m=0}^{\infty} \frac{(\frac{C}{B})_m (t)_m B^m}{(q)_m (At)_m} = (Bt)_\infty (A)_\infty \sum_{m=0}^{\infty} \frac{(\frac{C}{A})_m (t)_m A^m}{(q)_m (Bt)_m} .$$

Changing t to c, A to $\frac{aq}{c}$, C to $\frac{\lambda q}{c}$ and 9 to $\frac{bq}{c}$ in this we get (4).

<u>Corollary</u> . Identity (2) can be obtained from (4) on letting c $\longrightarrow$ co.

We now obtain two lemmas in order to prove (Theorem 2) the Ramanujan identity (10) on which we will base our proof of (6) (Theorem 3).

<u>Lemma 1</u>. 1% $|q|$, $|t|$ and $\left|\frac{ABt}{C}\right| < 1$, then the following

identity holds:

$$(16)\quad \sum_{n=0}^{\infty} \frac{(A)_n(B)_n t^n}{(q)_n(C)_n} = \frac{\left(\frac{ABt}{C}\right)_\infty}{(t)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{C}{A}\right)_n\left(\frac{C}{B}\right)_n}{(q)_n(C)_n}\left(\frac{ABt}{C}\right)^n .$$

This identity is also due to Heine [23].

Proof: We have seen in Theorem 1 that (9) can be reformulated as Heine's transformation (15). It is standard to prove (16) by iterating (15) twice and we have given the details in Appendix A.

Lemma 2. If $|q|<1$ and $|aq^m|<1$, then the following identity holds;

$$(17)\quad \sum_{n=0}^{\infty} \frac{(c)_n}{(q)_n(d)_n}(aq^m)^n = \frac{1}{(aq^m)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}}\left(\frac{d}{c}\right)_n (acq^m)^n}{(q)_n(d)_n}$$

Proof: Changing A to 0, 3 to c, C to d and t to aq^m in (16) we have the desired result.

Theorem 2. If $|q|<1$ and $|a|<1$, then the identity (10) holds.

Proof: We have

$$\frac{(a)_\infty}{(b)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n\left(\frac{b}{a}\right)_n a^n}{(q)_n(d)_n} = \frac{(a)_\infty\left(\frac{b}{a}\right)_\infty}{(b)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n a^n}{(q)_n(d)_n\left(\frac{bq^n}{a}\right)_\infty}$$

$$= \frac{(a)_\infty\left(\frac{b}{a}\right)_\infty}{(b)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n a^n}{(q)_n(d)_n} \sum_{m=0}^{\infty} \frac{\left(\frac{b}{a}\right)^m q^{nm}}{(q)_m} \quad \text{(by (8))}.$$

$$= \frac{(\frac{b}{a})_\infty}{(b)_\infty} \sum_{m=0}^{\infty} \frac{(a)_m (\frac{b}{a})^m (aq^m)_\infty}{(q)_m} \sum_{n=0}^{\infty} \frac{(c)_n (aq^m)^n}{(q)_n (d)_n}$$

$$= \frac{(\frac{b}{a})_\infty}{(b)_\infty} \sum_{m=0}^{\infty} \frac{(a)_m (\frac{b}{a})^m}{(q)_m} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} (\frac{d}{c})_n (acq^m)^n}{(q)_n (d)_n} \quad \text{(by lemma 2)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} (\frac{b}{a})_n (\frac{d}{c})_n (ac)^n}{(q)_n (b)_n (d)_n} .$$

Here the order of summation is interchanged and (8) is made use of.

Corollary 1. Putting $a=0$ and then changing d to $-aq$, b to $-bq$, c to $-\frac{\lambda}{b}$ in (10) we obtain an identity stated by Ramanujan in his 'lost' notebook [35], namely,

$$\sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{b})_n (bq)^n q^{\frac{n(n-1)}{2}}}{(q)_n (-aq)_n} = (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} \lambda^n (\frac{abq}{\lambda})_n}{(q)_n (-bq)_n (-aq)_n} .$$

Corollary 2. If $|q| < 1$, then the Ramanujan identity

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{n^2}}{(q)_n (aq)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n a^n (\frac{b}{a})_n q^{\frac{n(n+1)}{2}}}{(q)_n} ,$$

which is the same as (19) of Chapter II but for the notational differences, follows from (10) on putting $a=0=d$ and changing b to bq, c to $\frac{\beta}{b}$, b to a and lastly β to b.

Corollary 3. On putting $b = a$ in Corollary 2 above we obtain the identity

$$\frac{1}{(aq)_\infty} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n (aq)_n}, \quad |q| < 1,$$

due to Cauchy [15] and Jacobi [29]. This identity is also listed by Ramanujan in his works [34, Vol.II, p.193, Entry 3] and is the q-analogue of Kummer's hypergeometric transformation.

Theorem 3. If $|q| < 1$, then the identity (6) holds.

Proof: Changing a to $-\frac{aq}{c}$, b to $-bq$ and then c to $-\frac{\lambda}{b}$ in (10) we have

$$(18) \quad \hat{g}(a, \lambda, b, d, q) = \left(\frac{abq}{\lambda}\right)_\infty \sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{b})_n (-\frac{\lambda}{a})_n}{(q)_n (d)_n} \left(\frac{abq}{\lambda}\right)^n$$

where $\hat{g}$ is as in (6). Since the right side of (18) is symmetric in a and b, so must be the left side and hence the tneorem.

3. Some functional relations satisfied by $\tilde{g}$.

Lemma 3. If $|q|<1$ and $\left|\frac{aq}{c}\right|<1$ then $\tilde{g}$ given by (5) satisfies the following functional relations:

$$(19)\quad \tilde{g}(a,\lambda,b,c,q)-(1+\frac{aq}{c})\,\tilde{g}(aq,\lambda,b,c,q) = aq(1-\frac{1}{c})\,\tilde{g}(aq,\lambda q,bq,cq,q),$$

$$(20)\quad \tilde{g}(a,\lambda,b,c,q)-\tilde{g}(aq,\lambda,b,cq,q) = (\frac{\lambda}{c}+aq)\,\tilde{g}(aq,\lambda q,bq,cq,q),$$

$$(21)\quad \tilde{g}(a,\lambda,b,c,q)-\tilde{g}(a,\lambda q,b,c,q) = \lambda q(1-\frac{1}{c})\,\tilde{g}(aq,\lambda q^2,bq,cq,q).$$

If $\left|\frac{bq}{c}\right|<1$ in addition to the conditions above then,

$$(22)\quad \tilde{g}(a,\lambda,b,c,q)-\tilde{g}(a,\lambda,bq,cq,q) = (\frac{\lambda}{c}+bq)\,\tilde{g}(aq,\lambda q,bq,cq,q).$$

Proof: Since $(-\frac{\lambda}{a})_n - q^n(-\frac{\lambda}{aq})_n$ equals o if n is 0 and $(-\frac{\lambda}{a})_{n-1}(1-q^n)$ if $n\geq 1$ as is easily verified, we have left side of (19)

$$= (-bq)_\infty(-\frac{aq}{c})_\infty \sum_{n=0}^{\infty} \frac{(c)_n(-\frac{aq}{c})^n}{(q)_n(-bq)_n}\left[(-\frac{\lambda}{a})_n - q^n(-\frac{\lambda}{aq})_n\right]$$

$$= (-\frac{aq}{c})(1-c)(-bq^2)_\infty(-\frac{aq}{c})_\infty \sum_{n=1}^{\infty} \frac{(cq)_{n-1}(-\frac{\lambda}{a})_{n-1}(-\frac{aq}{c})^{n-1}}{(q)_{n-1}(-bq^2)_{n-1}}$$

$$= aq(1 - \frac{1}{c})\, \tilde{g}(aq, \lambda q, bq, cq, q)$$ proving (19).

The relations (20) and (21) follow on similar lines. That (20) implies (22) can be shown using (4).

4. Continued fraction developments (11) and (12) — their proofs

In this section, we establish generalizations of Ramanujan's q-continued fraction identities (I) and (II) dealt in Chapter II, using the results of Section 3.

Lemma 4. If $|q|<1$, $\left|\frac{aq}{c}\right|<1$ and $\left|\frac{bq}{c}\right|<1$ then $\tilde{g}$ satisfies

$$(23)\quad \tilde{g}(a,\lambda,b,c,q) = \left(1+\frac{aq}{c}\right)\tilde{g}(aq,\lambda q,b,c,q) + (aq+\lambda q)\left(1-\frac{1}{c}\right)\tilde{g}(aq,\lambda q^2,bq,cq,q),$$

$$(24)\quad \tilde{g}(a,\lambda,b,c,q) = \tilde{g}(a,\lambda q,bq,cq,q) + (\lambda q+bq)\,\tilde{g}(aq,\lambda q^2,bq,cq,q).$$

Proof: Changing λ to λq in (19) and adding it to (21) we get (23). Changing λ to λq in (22) and adding it to (21) we get (24).

Theorem 4. If $|q|<1$ and $\left|\frac{aq}{c}\right|<1$, and $\left|\frac{bq}{c}\right|<1$, then the identity (11) holds

Proof: Changing a to aq^n, λ to λq^{2n}, b to by" and c to cq^n in (23) and changing a to aq^{n+1}, λ to λq^{2n+1}, b to by" and c to cq^n in (24) we can write (23) and (24) respectively as

$$Q_n \equiv \frac{\tilde{g}(aq^n,\lambda q^{2n},bq^n,cq^n,q)}{\tilde{g}(aq^{n+1},\lambda q^{2n+1},bq^n,cq^n q)} = \left(1+\frac{aq}{c}\right) + \frac{(aq^{n-i-l}+\lambda q^{2n+1})\left(1-\frac{1}{n}\right)}{Q'n}$$

$$Q'_n \equiv \frac{\tilde{g}(aq^{n+1}, \lambda q^{2n+1}, bq^n, cq^n, q)}{\tilde{g}(aq^{n+1}, \lambda q^{2n+2}, bq^{n+1}, cq^{n+1}, q)} = 1 + \frac{bq^{n+1} + \lambda q^{2n+2}}{Q_{n+1}}$$

Iterating the last two identities alternately with n= 0,1,2...

we have (11).

Lemma 5. If $|q| < 1$, $\left|\frac{aq}{c}\right| < 1$, and $\left|\frac{bq}{c}\right| < 1$, then $\tilde{g}$ satisfies

$$(25) \quad \tilde{g}(aq, \lambda, b, c, q) = \frac{(1+bq-aq) + \frac{1}{c}(a+\lambda+aq)}{(1+\frac{aq}{c})}\tilde{g}(aq, \lambda q, bq, cq, q)$$

$$+ \frac{(aq+\lambda q)(1-\frac{1}{cq})}{(1+\frac{aq}{c})}\tilde{g}(aq, \lambda q^2, bq^2, cq^2, q)$$

Proof: Changing c to cq, λ to λq, b to bq in (19) on the one hand, b to bq, c to cq in (21) on the other, and then taking the negative of (19) and adding these three equations to (22) we get (25).

Theorem 5. If $|q| < 1$ and $\left|\frac{aq}{c}\right| < 1$ and $\left|\frac{bq}{c}\right| < 1$, then the identity (12) holds.

Proof: Identity (2.3) can be written as

$$(26) \quad \frac{\tilde{g}(aq, \lambda q, b, c, q)}{\tilde{g}(a, \lambda, b, c, q)} = \cfrac{1}{1+\frac{aq}{c}+\cfrac{(aq+\lambda q)(1-\frac{1}{c})}{\cfrac{\tilde{g}(aq, \lambda q, b, c, q)}{\tilde{g}(aq, \lambda q^2, bq, cq, q)}}} .$$

Changing λ to λq^{n+1}, b to bq^n, c to cq^n (25) can be written as

$$s_n \equiv \frac{\tilde{g}(aq, \lambda q^{n+1}, bq^n, cq^n, q)}{\tilde{g}(aq, \lambda q^{n+2}, bq^{n+1}, cq^{n+1}, q)}$$

$$= \frac{(1-aq+bq^{n+1}) + \frac{1}{cq^n}(a+aq+\lambda q^{n+1})}{\left(1+\frac{aq}{cq^n}\right)} + \frac{(aq+\lambda q^{n+2})\left(1-\frac{1}{cq^{n+1}}\right)}{\left(1+\frac{aq}{cq^n}\right)s_{n+1}}$$

Iterating this with n = 0,1,2... and using (26) we have (12).

5. An elementary proof of the continued fraction identity (14)

We first make some notations. For $|q|<1$ and $|a|<1$ let

$$N_0 = \sum_{n=0}^{\infty} \frac{(\frac{b}{a})_{2n+1}\, a^{2n+1}}{(q)_{2n+1}}, \quad D_0 = \sum_{n=0}^{\infty} \frac{(\frac{b}{a})_{2n}\, a^{2n}}{(q)_{2n}},$$

$$N_1 = D_0, \quad D_1 = \sum_{n=0}^{\infty} \frac{(\frac{bq}{a})_{2n}\, a^{2n}}{(q^2,q)_{2n}} = \frac{(1-q)}{(a-b)} N_0,$$

and for $m>1$

$$N_m = (1-q^{2m-2})\, D_{m-1}, \quad D_m = \sum_{n=0}^{\infty} \frac{(\frac{bq^m}{a})_{2n}}{(q^{2m},q)_{2n}} \prod_{k=1}^{m-1} (1-q^{2n+2k}) a^{2n}.$$

Let $A_{m,n}$ and $B_{m,n}$ denote the coefficients of a^{2n} in N_m and D_m respectively.

Lemma 6. For $m \geq 1$,

$$(27) \quad A_{m,n} - B_{m,n} = \begin{cases} 0, & \text{if } n=0 \\ \dfrac{q^{m-1}(1-\frac{bq^m}{a})(q^m-\frac{b}{a})\, B_{m+1,n-1}}{(1-q^{2m-1})(1-q^{2m})(1-q^{2m+1})}, & \text{if } n>0. \end{cases}$$

Proof. The cases m= 1 and 2 are quite easily verified. For $m>2$, we have, from the definitions of $A_{m,n}$ and $B_{m,n}$,

$$A_{m,0} - B_{m,0} = (1-q^{2m-2}) \prod_{k=1}^{m-2} (1-q^{2k}) - \prod_{k=1}^{m-1} (1-q^{2k}) = 0$$

and for $n > 0$, on removing common factors,

$$A_{m,n}-B_{m,n} = \frac{\left(\frac{bq^m}{a}\right)_{2n-1}}{(q^{2m-1})_{2n+1}} \prod_{k=1}^{m-1} (1-q^{2n+2k}) \left\{ (1-q^{2n+2m-1})\left(1-\frac{b}{a}q^{m-1}\right) - \left(1-\frac{b}{a}q^{2n+m-1}\right)(1-q^{2m-1}) \right\}.$$

Equality (27) now readily follows.

Lemma 7. If $|q| < 1$ and $|a| Al.$, then for $m \geq 1$

$$(28) \quad (1-q^{2m-1}) \frac{N_m}{D_m} = (1-q^{2m-1}) + \frac{q^{m-1}(a-bq^m)(aq^m-b)}{(1-q^{2m+1})\frac{N_{m+1}}{D_{m+1}}}.$$

Proof : Multiplying both sides of (27) by a^{2n} and then summing over n from 0 to ∞ we get that

$$N_m - D_m = \frac{(q^{m-1})(a-bq^m)(aq^m-b)}{(1-q^{2m-1})(1-q^{2m})(1-q^{2m+1})} D_{m+1}$$

Or

$$(1-q^{2m-1})\left(\frac{N_m}{D_m}-1\right) = \frac{q^{m-1}(a-bq^m)(aq^m-b)}{(1-q^{2m+1})\frac{(1-q^{2m})D_m}{D_{m+1}}}$$

from which the required result follows since $(1-q^{2m})D_m = N_{m+1}$ by definition.

Theorem 6. If $|q| < 1$ and $|a| < 1$, then

$$(14)\quad \frac{(-a)_\infty (b)_\infty - (a)_\infty (-b)_\infty}{(-a)_\infty (b)_\infty + (a)_\infty (-b)_\infty} = \frac{a-b}{1-q+}\ \frac{(a-bq)(aq-b)}{1-q^3+}\ \frac{q(a-bq^2)(aq^2-b)}{1-q^5+}\ \cdots\ \frac{q^{n-1}(a-bq^n)(aq^n-b)}{1-q^{2n+1}+}\ \cdots$$

Proof: On using (8) we have

$$(29)\quad \frac{(-a)_\infty (b)_\infty - (a)_\infty (-b)_\infty}{(-a)_\infty (b)_\infty + (a)_\infty (-b)_\infty} = \frac{\frac{(b)_\infty}{(a)_\infty} - \frac{(-b)_\infty}{(-a)_\infty}}{\frac{(b)_\infty}{(a)_\infty} + \frac{(-b)_\infty}{(-a)_\infty}}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{(\frac{b}{a})_{2n+1}}{(q)_{2n+1}} a^{2n+1}}{\sum_{n=0}^{\infty} \frac{(\frac{b}{a})_{2n}\ a^{2n}}{(q)_{2n}}}$$

$$= \frac{(a-b)}{(1-q)\ \frac{N_1}{D_1}}$$

Iterating (28) with $m = 1, 2 \ldots$ and using (29) we have the required result. The convergence of continued fraction follows since $\frac{N_m}{D_m} = (1-q^{2m-2}) \frac{D_{m-1}}{D_m} \longrightarrow 1$ as $m \longrightarrow \infty$ under the conditions $|q| < 1$ and $|a| < 1$. Berndt (private communication/ incorporated in [1]), has recently obtained another proof of (14) employing a continued fraction of Heine [23].