Chapter IV*

ON A 'REMARKABLE FORMULA' OF SRINIVASA RAMANUJAN

1. Introduction

The following identity (1) stated without proof by Srinivasa Ramanujan [34, Vol.II, p.196, Entry 17] has been described by Hardy [21, p.222] as a 'remarkable formula' :

$$(1)\quad \frac{(-qz,q^2)_\infty\,(-\frac{q}{z},q^2)_\infty}{(-\alpha qz,q^2)_\infty\,(-\frac{\beta q}{z},q^2)_\infty}$$

$$= \frac{(\alpha q^2,q^2)_\infty\,(\beta q^2,q^2)_\infty}{(q^2,q^2)_\infty\,(\alpha\beta q^2,q^2)_\infty}\sum_{-\infty}^{\infty}\frac{(\frac{1}{\alpha},q^2)_k\,(-\alpha qz)^k}{(\beta q^2,q^2)_k}\,.$$

Here, as usual, $|q|<1$,

$$(c)_\infty = (c,q)_\infty = \prod_{k=0}^{\infty}(1-cq^k)$$

and

$$(c)_k = (c,q)_k = \frac{(c)_\infty}{(cq^k)_\infty}\quad (k\text{: any integer}).$$

We devote the present short chapter on this single formula because of its basic importance in literature. It has many applications in the theory of q-series. For instance, putting $\beta=1$ in (1) we obtain the q-binomial theorem of Euler.

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* Reference [1] is partly based on this chapter.

$\alpha=\beta=0$ in (1) gives the Jacobi's triple product identity well-known in the theory of theta-functions [22, pp.282-283] [29][45]. While we discuss many applications of (1) in a later chapter, we give below a Proof of (1) without using the q-binomial theorem.

Denoting the left side of (1) by $f(z)$, we seek its power series expansion $\sum_{-\infty}^{\infty} c_k z^k$ in the region of analyticity $|\beta q| < |z| < \frac{1}{|\alpha q^{-1}|}$. The coefficients c_k are at once found, up to a factor c_o, on using a functional relation that f(z) is seen to satisfy trivially. The constant c_o is then found by equating the limits, as $z \longrightarrow -\frac{1}{\alpha q}$, of $(1+\alpha qz) \sum_{-\infty}^{\infty} c_k z^k$ and $(1+\alpha qz)\, f(z)$.

Several other proofs of (1) such as those of Andrews [2] [3], Andrews and Askey [8], Askey [9], Hahn [20], Ismail [27] and Jackson [28] can be found in literature. Many of these proofs depend on the q-binomial theorem. A direct proof would perhaps put (1) in a proper perspective in literature.

2. Proof of the 'remarkable formula'

Theorem 1. If $|q|<1$ and $|\alpha\beta q^2|<1$ then the identity (1) holds in the region $|\beta q|<|z|<\frac{1}{|\alpha q|}$.

Proof. Let us define

$$(2)\quad f(z) = \frac{(-qz,q^2)_\infty\,(-\frac{q}{z},q^2)}{(-\alpha qz,q^2)_\infty\,(-\frac{\beta q}{z},q^2)}$$

and seek the expansion

$$(3)\quad f(z) = \sum_{-\infty}^{\infty} c_k z^k$$

in the region of analyticity $|\beta q|<|z|<\frac{1}{|\alpha q|}$.

We can easily see from (2) that

$$(4)\quad (\beta+qz)\,f(q^2z) = \frac{(-qz,q^2)_\infty\,(-\frac{q}{z},q^2)_\infty}{(-\alpha q^3z,q^2)_\infty\,(-\frac{\beta q}{z},q^2)_\infty} = (1+\alpha qz)\,f(z).$$

From (3), we have the expansion

$$(5)\quad (\beta+qz)\,f(q^2z) = \sum_{-\infty}^{\infty}(c_k q\beta + c_{k-1})\,q^{2k-1}z^k$$

valid at least in $\left|\frac{\beta}{q}\right|<|z|<\frac{1}{|\alpha q^3|}$. It is in fact valid in the bigger annulus $|\beta q|<|z|<\frac{1}{|\alpha q^3|}$ since, as seen by the first of the equations (4), the left side of (5) is analytic there. Similarly, (3) and (4) imply the expansion

$$(6)\quad (1+\alpha qz)\ f(z) = \sum_{-\infty}^{\infty} (c_k + \alpha q\ c_{k-1})\ z^k$$

valid in the same annulus $|\beta q| < |z| < \frac{1}{|\alpha q|}$. From (4), we can equate the coefficients of like powers of z in the right side; of (5) and (6). Hence we get

$$c_k = \frac{-\alpha q(1-\frac{q^{2k-2}}{\alpha})}{(1-\beta q^{2k})} c_{k-1}, \quad (k:\ \text{any integer}),$$

iterating which we have,

$$(7)\quad c_k = \frac{(-\alpha q)^k (\frac{1}{\alpha}, q^2)_k}{(\beta q^2, q^2)_k} c_0 \quad (k:\ \text{any integer}).$$

It is clear from (7) that, since $|q| < 1$, the series of positive powers of z and the series of negative Towers of z in the right side of (3) are respectively analytic in $|z| < \frac{1}{|\alpha q|}$ and $|z| > |\beta q|$. Further, by (2), f(z) has a simple pole at $z = -\frac{1}{\alpha q}$. Hence, by (3),

$$\lim_{z \to -\frac{1}{\alpha q}} (1+\alpha qz)\ f(z) = \lim_{z \to -\frac{1}{\alpha q}} (1+\alpha qz) \sum_{-\infty}^{\infty}{}' c_k z^k$$

$$= \lim_{z \to -\frac{1}{\alpha q}} (1+\alpha qz) \sum_{k=0}^{\infty} c_k z^k$$

$$= \lim_{z \to -\frac{1}{\alpha q}} \lim_{n \to \infty} \left\{ c_0 + \sum_{k=1}^{n} \left[c_k + (\alpha q) c_{k-1} \right] z^k \right\}$$

$$= \lim_{n \to \infty} \left[c_0 + \sum_{k=1}^{n} (-1)^k \left\{ \frac{c_k}{(\alpha q)^k} + \frac{c_{k-1}}{(\alpha q)^{k-1}} \right\} \right]$$

$$= \lim_{n \to \infty} \frac{(-1)^n c_n}{(\alpha q)^n}$$

$$(8) \qquad = c_0 \frac{(\frac{1}{\alpha}, q^2)_\infty}{(\beta q^2, q^2)_\infty} \quad \text{(using (7)),}$$

where the inversion in limit processes is justified because the sequence of partial sums of $(1+\alpha qz) \sum_{k=0}^{\infty} c_k z^k$ converges uniformly in a sufficiently small neighbourhood of $z = -\frac{1}{\alpha q}$.

Using (2) in (8) we deduce

$$(9) \quad c_0 = \frac{(\alpha q^2, q^2)_\infty (\beta q^2, q^2)_\infty}{(q^2, q^2)_\infty (\alpha \beta q^2, q^2)_\infty}.$$

We complete the proof on substituting (7) and (9) in (3).

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In the next two chapters we study parts of the theory of theta-functions and q-series as developed by Ramanujan and we will find in identity (1) a highly useful tool.