Chapter V*

ON SOME IDENTITIES DEDUCIBLE DIRECTLY FROM RAMANUJAN'S DEFINITION OF THETA FUNCTION.

1. Introduction

In this chapter we obtain a few interesting identities found in Ramanujan's works [34, Vol. II, Chapter 16] which are directly deducible from his definition of theta-function:

$$(1)\quad f(a,b) = 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n+b^n) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \qquad |ab| < 1.$$

This is nothing but the theta-function

$$\vartheta_3(z,t) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz) = \sum_{-\infty}^{\infty} q^{n^2} \cos(2nz),\quad q = e^{i\pi t}, \qquad \mathrm{Im}\, t > 0,$$

in the classical notation.

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* Reference [1] is partly based on this chapter.

2. Some interesting identities

To prove the first two of our theorems we find it convenient to obtain the following simple lemmas.

Lemma 1. If n is any integer and

(2) $$\sigma(n) = \frac{n(n+1)}{2}$$

then

(3) $$\sigma(n) - \sigma(-n) = n,$$

(4) $$\sigma(m+n) - \sigma(m) = \sigma(n) + mn$$

and

(5) $$\sigma(m)\,\sigma(n) + \sigma(-m)\,\sigma(-n) = \sigma(mn).$$

Proof: The Lemma follows by an easy verification and we omit the details.

Lemma 2. If a and b are complex numbers and if

(6) $$U_n = a^{\sigma(n)} b^{\sigma(-n)}$$

where $\sigma(n)$ is as in (2), then

(7) $$U_{m+n} = (ab)^{mn} U_m U_n,$$

(8) $$U_{m+n} = W_m U_n$$

where

(9) $$W_m = \left\{a(ab)^n\right\}^{\sigma(m)} \left\{b(ab)^{-n}\right\}^{\sigma(-m)}$$

and

$$(10)\quad \left(\frac{U_{m+n}}{U_m}\right)^{\sigma(k)} \left(\frac{U_{m-n}}{U_m}\right)^{\sigma(-k)} = \frac{U_{m+nk}}{U_m}.$$

Proof: Substituting for the U's from the definition (6) we have

$$\frac{U_{m+n}}{U_m U_n} = a^{\sigma(m+n)-\sigma(m)-\sigma(n)}\, b^{\sigma(-m-n)-\sigma(-m)-\sigma(-n)}$$

$$= a^{mn}\, b^{mn}.$$

Hence (7) is proved.

Now, by the definition (9)

$$W_m = \left(a(ab)^n\right)^{\sigma(m)} \left(b(ab)^{-n}\right)^{\sigma(-m)}$$

$$= a^{\sigma(m)}\, b^{\sigma(-m)}\, (ab)^{n(\sigma(m)-\sigma(-m))}$$

$$= U_m (ab)^{mn}, \text{ by (6) and (3).}$$

Using this in (7) gives (8).

Lastly, substituting for the U's from the definition (6) we can write

$$(11)\quad \text{left side of (10)} = a^{\sigma'(m,n,k)}\, b^{\sigma'(-m,-n,k)}$$

where

$$\sigma'(m,n,k) = \left[\sigma(m+n)-\sigma(m)\right]\sigma(k) + \left[\sigma(m-n)-\sigma(m)\right]\sigma(-k)$$

$$= \left[\sigma(n)+mn\right]\sigma(k) + \left[\sigma(-n)-mn\right]\sigma(-k), \text{ by (4)}$$

$$= \sigma(nk) + mnk, \text{ by (5) and (3)}$$

$$= \sigma(m+nk) - \sigma(m), \text{ by (4) again.}$$

Substituting this in (11) we get (10) in view of (6).

Theorem 1. If $|ab| < 1$ and if n is any integer then

$$(12) \quad f(a,b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f\left(a(ab)^{n}, b(ab)^{-n}\right)$$

where $f(a,b)$ is as in (1).

Proof: We first observe that, using (6) in (1), we can write

$$(13) \quad f(a,b) = \sum_{-\infty}^{\infty} U_m .$$

Now, in the notations of Lemma 1 and 2, we have

$$a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f\left(a(ab)^{n}, b(ab)^{-n}\right)$$

$$= U_n \sum_{-\infty}^{\infty} W_k$$

$$= U_n \sum_{k=-\infty}^{\infty} \left(\frac{U_{n+k}}{U_n}\right) \text{ by (8)}$$

$$= \sum_{m=-\infty}^{\infty} U_m \quad \text{(on changing } n+k \text{ to } m)$$

$$= f(a,b) \quad \text{(by (13))}.$$

Hence the theorem.

When $n = 1$, (12) can be written in the classical notations as

$$\vartheta_3(z+\pi t, t) = q^{-1} e^{-2iz} \vartheta_3(z,t)$$

which is the well-known quasi-periodicity of the theta-function.

We now prove another identity of Ramanujan [34, Vol.II, p.200, Entry 31] and illustrate two corollaries also stated by him. We however furnish a general formulation of his corrollaries. It has been observed by Berndt (private communication, incorporated in [1]) that Ramanujan identity (14) below can be restated with the range of summation $\{-\alpha(n) \dots \alpha(n)\}$ on the right hand side replaced by any residue class mod n.

Theorem 2. If $|ab| < 1$ and n, any positive integer, then

$$(14) \quad f(a,b) = \sum_{-\alpha(n)}^{\alpha(n)} U_m \, f\left(\frac{U_{n+m}}{U_m}, \frac{U_{m-n}}{U_m} \right) + R_n$$

$$(15) \quad \text{where } \alpha(n) = \begin{cases} \frac{n-1}{2}, \text{ if } n \text{ is odd,} \\ \frac{n-2}{2}, \text{ if } n \text{ is even,} \end{cases}$$

and

$$(16) \quad R_n = \begin{cases} 0, \text{ if } n \text{ is odd,} \\ U_{\frac{n}{2}} \, f\left(\frac{U_{\frac{3n}{2}}}{U_{\frac{n}{2}}}, \frac{U_{-\frac{n}{2}}}{U_{\frac{n}{2}}} \right), \text{ if } n \text{ is even.} \end{cases}$$

f and U_m are as in (1) and (6) respectively.

Proof: If $\alpha(n)$ and R_n are as in (15) and (16) and if we put

$$\gamma_k(n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ U_{nk+\frac{n}{2}}, & \text{if } n \text{ is even,} \end{cases}$$

We have, upon substituting (6) in (1),

$$f(a,b) = \sum_{-\infty}^{\infty} U_k$$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\alpha(n)}^{\alpha(n)} U_{nk+m} + \gamma_k(n) \right)$$

$$= \sum_{m=-\alpha(n)}^{\alpha(n)} \sum_{k=-\infty}^{\infty} U_{nk+m} + \sum_{k=-\infty}^{\infty} \gamma_k(n)$$

$$= \sum_{m=-\alpha(n)}^{\alpha(n)} U_m \, f\left(\frac{U_{m+n}}{U_m}, \frac{U_{m-n}}{U_m} \right) + R_n$$

Here we have used (10) and the definition (1) of f. This completes the proof of the theorem.

Ramanujan also uses the following functions which are restrictions of f:

(17) $\phi(q) = f(q,q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}$,

$$(18) \quad \psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}},$$

and

$$(19) \quad f(-q) = f(-q,-q^2) = \sum_{-\infty}^{\infty} -1)^n q^{\frac{n(3n-1)}{2}}$$

In the above functions $|q| < 1$.

Quite frequently in our work, we need the following properties which we state as

Lemma 1. If $|q| < 1$, then

$$(23) \quad f(1,q) = 2 f(q,q^3) = 2\psi(q)$$

and

$$(21 \quad f(-1,q = 0.$$

Proof: $f(1,q) = 1 + \sum_{n=1}^{\infty} q^{\frac{n(n-1)}{2}} + \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}}$

$$= 2 + \sum_{n=2}^{\infty} q^{\frac{n(n-1)}{2}} + \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}}$$

$$= 2\left(1 + \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}}\right)$$ (Changing n to n+1 in the first summation)

$$= 2\left[1 + \sum_{n=\text{even} > 0} q^{\frac{n(n+1)}{2}} + \sum_{n=\text{odd} > 0} q^{\frac{n(n+1)}{2}}\right]$$

$$= 2\left[1 + \sum_{n=1}^{\infty} q^{n(2n+1)} + \sum_{n=1}^{\infty} q^{n(2n-1)}\right]$$

$$= 2\left[1 + \sum_{n=1}^{\infty} q^{\frac{n(n-1)}{2}} (q^3)^{\frac{n(n+1)}{2}} + \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}} (q^3)^{\frac{n(n-1)}{2}}\right]$$

$$= 2\ f(q,q^3)$$

$$= 2\psi(q).$$

<u>roof of (21)</u> : $f(-1,q) = 1 + \sum_{n=1}^{\infty} (-q)^{\frac{n(n-1)}{2}} \left\{(-1)^n + (q)^n\right\}$

$$= \sum_{n=2}^{\infty} (-1)^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} q^{\frac{n(n+1)}{2}}$$

$= 0$ (on changing n to n+1 in the first summation).

<u>Corollary 1</u>. If $|q| < 1$ then,

$$(22)\ \phi(q) = \phi(q^{n^2}) + 2\sum_{m=1}^{\alpha(n)} q^{m^2} f\left(q^{n(n-2m)}, q^{n(n+2m)}\right)$$

$$+ \ 0 \text{ or } 2\, q^{\frac{n^2}{4}} \psi(q^{2n^2})$$

according as n is odd or even,

and

$$(23)\ \psi(q) = \frac{1}{2}\sum_{m=-\alpha(n)}^{\alpha(n)} q^{\frac{m(m-1)}{2}} f\left(q^{\frac{n(n-2m+1)}{2}-}, q^{\frac{n(n+2m-1)}{2}-}\right)$$

$$+ \ 0 \text{ or } \frac{1}{2}\, q^{\frac{n(n-2)}{8}} f\left(q^{\frac{n}{2}}, q^{\frac{n(2n-1)}{2}}\right)$$

according as n is odd or even.

$$(24)\quad \psi(q) = \sum_{m=-\alpha(n)}^{\alpha(n)} q^{m(2m-1)} f(q^{n(2n-4m+1)}, q^{n(2n+4m-1)})$$

$$+\ 0 \text{ or } q^{\frac{n(n-1)}{2}} f\left(q^{n}, q^{n(4n-1)}\right)$$

according as n is odd or even.

<u>Proof</u>: Putting $a = q = b$ in (14) and using (20) we get (22).

Putting $a = 1$, $b=q$ in (14) and using (20) we get (23).

Putting $a=q$, $b = q^3$ in (14) we get (24).

<u>Corollary 2</u>. (Ramanujan) If $|q| < 1$ then,

$$(25)\quad \phi(q) = \phi(q^9) + 2q\, f(q^3, q^{15})$$

$$(25)\quad = \phi(q^{25}) + 2q\, f(q^{15}, q^{35}) + 2q^4 f(q^5, q^{45})$$

$$(27)\quad \psi(q) = f(q^3, q^6) + q\, \psi(q^9)$$

$$(28)\quad = f(q^6, q^{10}) + q\, f(q^2, q^{14})$$

$$(29)\quad = f(q^{10}, q^{15}) + q\, f(q^5, q^{20}) + q^3 \psi(q^{25})$$

$$(30)\quad = f(q^{15}, q^{21}) + q\psi(q^9) + q^3 f(q^3, q^{33}).$$

<u>Proof</u>: For (25) put a=b=q and $n = 3$ in (14)

For (26) put a=b=q and $n = 5$ in (14).

For (27) put $a = 1$, $b = q$ and $n = 3$ in (14) and use (20).

For (23) put a=q, $b = q^3$ and $n =$ 2 in (14).

For (29) put $a = 1$, $b = q$ and $n = 5$ in (14) and use (20).

For (30) put $a = q$, $b = q^3$ and $n = 3$ in (14).

A particular case of the second of the two Ramanujan identities (31) and (32) [34, Vol.II, p.203, Entry 36] proved in next theorem has been applied by Hardy [21, p.223] in discussing Ramanujan's approach to derive modular equations of degree 3. Hardy also sketches a proof of (34) We give here details of the proof of (31) to elucidate how it simply follows from Ramanujan's definition (1). Proof of (32) is similar. As an application of (31) and (32) we derive a few more Ramanujan identities (33) - (38). Watson [43] has obtained some general formulae which are essentially due to Schroter [41, pp.163-167] on applying (12), (14), (31) and (32) collectively. Berndt ('private communication; incorporated in [1]) has thereby proved certain Ramanujan identities concerning f and ψ [34, Vol.II, p.204, corollary under Entry 37 and 38(iv)]. We deduce a few other identities also due to Ramanujan [34].

Theorem 3: If $|ab| < 1$, $|cd| < 1$ and $p = \frac{ab}{cd}$, then

$$(31)\quad \frac{1}{2}\left\{ f(a,b)\, f(c,d) + f(-a,-b)\, f(-c,-d) \right\}$$

$$= \sum_{-\infty}^{\infty} (ad)^{\frac{n(n+1)}{2}} (bc)^{\frac{n(n-1)}{2}} f\left(acp^n, \frac{bd}{p^n}\right)$$

and

$$(32)\ \tfrac{1}{2}{}_{\mathrm{I}}\ f(a,b)\ f(c,d) - f(-a,-b)\ f(-c,-d)\ {}_{3}$$

$$= a \sum_{-\infty}^{\infty} a^{2n} (ad)^{\frac{n(n-1)}{2}} (bc)^{\frac{n(n+1)}{2}} f\left(\frac{c}{ap^{n}}, \frac{ap^{n}}{c}\, abcd\right)$$

where f is as defined in (1).

Remark: If $ab = cd$, then these become respectively

$$(33)\ f(a,b)\ f(c,d) + f(-a,-b)\ f(-c,-d) = 2\ f(ac,bd)\ f(ad,bc)$$

and

$$(34)\ f(a,b)\ f(c,d) - f(-a,-b)\ f(-c,-d) = 2a\ f\left(\frac{b}{c}, \frac{c}{b}\, abcd\right) f\left(\frac{b}{d}, \frac{d}{b}\, abcd\right).$$

These and some of their special cases such as

(35) $\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2)$ (put $a=b=c=d=q$ in (33)),

(36) $\phi^2(q) - \phi^2(-q) = 8q\,\psi^2(q^4)$ (put $a=b=c=d=q$ in (34) and use (20)),

(37) $f^2(a,b) + f^2(-a,-b) = 2\ f(a^2,b^2)\ \phi(ab)$, (put $c=a$, $d=b$ in (33))

and

(38) $f^2(a,b) - f^2(-a,-b) = 4a\ f\left(\frac{b}{a}, \frac{a}{b}\, a^2b^2\right)\psi(a^2b^2)$,

(put $c=a$, $d=b$ in (34) and use (20)),

are also due to Ramanujan. Proof of identity (38) using

Landen's quadratic transforms has been given by Tannery and Molk [21, p.223; 41(ii), pp.114-119, p.268 (Table XLVII)]. We will consider an application of (33) and (34) in Theorem 5 of the next chapter obtaining an interesting Ramanujan-identity involving several parameters.

Proof of Theorem 3. We can write left side of (31) as

$$= \sum_{\substack{m=-\infty \\ m+n \text{ even}}}^{\infty} \sum_{n=-\infty}^{\infty} (cd)^{\frac{m^2+n^2-m-n}{2}} a^n c^m p^{\frac{n(n-1)}{2}}$$

$$= \sum_{\lambda=-\infty}^{\infty} \sum_{\mu=-\infty}^{\infty} (cd)^{\lambda^2+\mu^2-\lambda} a^{\lambda-\mu} c^{\lambda+\mu} p^{\frac{\lambda^2+\mu^2-\lambda+\mu}{2}-\lambda\mu}$$

(Since m & n must be of the form $m+n = 2\lambda$, $m-n = 2\mu$)

$$= \sum_{\mu=-\infty}^{\infty} (ad)^{\frac{\mu(\mu-1)}{2}} (bc)^{\frac{\mu(\mu+1)}{2}} \sum_{\lambda=-\infty}^{\infty} \left(\frac{ac}{p^{\mu}}\right)^{\frac{\lambda(\lambda+1)}{2}} (bdp^{u})^{\frac{\lambda(\lambda-1)}{2}}$$

On changing μ to $-\mu$ we yet the required result.

In the following corollary 3 we obtain further Ramanujan identities [34, Vol.II, pp.203-204, Entry 37].

Corollary 3. If $|a| < 1$ and $|b| < 1$, then

$$(39)\quad \frac{1}{2}\left\{ \phi(a)\,\phi(b) + \phi(-a)\,\phi(-b) \right\}$$

$$= \phi(ab) + 2\sum_{n=1}^{\infty} (ab)^{n^2} f\left(ab\,\frac{a^{2n}}{b^{2n}},\ ab\,\frac{b^{2n}}{a^{2n}}\right),$$

$$(40)\quad \frac{1}{2}\left\{ \phi(a)\,\phi(b) - \phi(-a)\,\phi(-b) \right\}$$

$$= 2\sum_{n=1}^{\infty} a^{n^2}\, b^{(n-1)^2}\, f\left(\frac{b^{2n-1}}{a^{2n-1}},\ (ab)^2\,\frac{a^{2n-1}}{b^{2n-1}}\right)$$

and

$$(41)\ \psi(a)\,\psi(b) = \psi(ab) + \sum_{n=1}^{\infty} a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n-1)}{2}}\, f\left(\frac{b^n}{a^n},\ ab\,\frac{a^n}{b^n}\right).$$

Proof : For (39) put $b = a$, $d = c$ and then change c to b in (31). To get, (40), put $b = a$, $d = c$ and then change c to b in (32) and use the identity

$$f\left(\frac{a^{2n-1}}{b^{2n-1}},(ab)^2\,\frac{b^{2n-1}}{a^{2n-1}}\right) = \frac{a^{2n-1}}{b^{2n-1}}\, f\left(\frac{b^{2n-1}}{a^{2n-1}},(ab)^2\,\frac{a^{2n-1}}{b^{2n-1}}\right)$$

which is a consequence of a particular case of (12), namely,

$$(42)\ f(A,B) = A\ f\left(A(AB),\ B(AB)^{-1}\right).$$

On putting $a=c=1$ and changing d to a in (31) we get

$$(43)\ \frac{1}{2}\left\{ f(1,b)\ f(1,a) + f(-1,-b)\ f(-1,-a)\right\}$$

$$= \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n-1)}{2}}\, f\left(\frac{b^n}{a^n},ab\,\frac{a^n}{b^n}\right).$$

Now, putting $A = \frac{a''}{b^n}$ and $B = ab\frac{b^n}{a^n}$ in (42) we have

$$(44)\quad f\left(\frac{a^n}{b^n},\ ab\,\frac{b^n}{a^n}\right) = \frac{a^n}{b^n}\ f\left(\frac{b^n}{a^n},ab\,\frac{a^n}{b^n}\right).$$

Using (20),(21) and (44) in (43) we have the required result.

Chapter VI*

ON SOME APPLICATIONS OF RAMANUJAN'S 'REMARKABLE FORMULA' IN THE THEORY OF q-SERIES

1. Introduction

We now return to the Ramanujan identity:

$$(1)\quad \left\{\frac{(-qz,q^2)_\infty\,(-\frac{q}{z},q^2)_\infty}{(-\alpha qz,q^2)_\infty\,(-\frac{\beta q}{z},q^2)_\infty}\right\}\cdot\left\{\frac{(q^2,q^2)_\infty\,(\alpha\beta q^2,q^2)_\infty}{(\alpha q^2,q^2)_\infty\,(\beta q^2,q^2)_\infty}\right\}$$

$$= 1+\sum_{n=1}^{\infty}\frac{(-1)^n(\alpha q)^n(\frac{1}{\alpha},q^2)_n}{(\beta q^2,q^2)_n}z^n+\sum_{n=1}^{\infty}\frac{(-1)^n(\beta q)^n(\frac{1}{\beta},q^2)_n}{(\alpha q^2,q^2)_n}z^{-n}$$

where $|q|<1$ and $|\beta q|<|z|<\frac{1}{|\alpha q|}$.

We gave a proof of this in Chapter IV and presently we apply it in order to prove a few more q-series identities stated in Ramanujan's works [34, Vol.II, Chapter 16].

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* Reference [1] is partly based on this Chapter.